MATH 213 Notes

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From lectures by David Wang

Contents

1 Method of Undetermined Coefficients

This method is used to guess the form of the particular solution to a nonhomogeneous solution.

Function of x	Form of y
e^{kx}	ke^{kx}
x^n	$k_1x^n + k_2x^{n-1} + ... + k_{n-1}x + k_n$
$\cos \omega x, \sin \omega x$	$A \cos \omega x + B \sin \omega x$
$e^{kx} \cos \omega x, e^{kx} \sin \omega x$	$e^x(A \cos \omega x + B \sin \omega x)$

If what we have is already in the solution, multiply it by x.

2 Second Order DE

2.1 Solving

Consider the second order homogenous case $y'' + ay' + by = 0$. Assume a solution is in the form $y = e^{\lambda x}$. Then derivatives are $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. So we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

Example 2.1

$$\begin{aligned} y'' + y' - 2y &= 0 \\ (\lambda^2 + \lambda - 2) &= 0 \\ (\lambda + 2)(\lambda - 1) &= 0 \end{aligned}$$

So $\lambda_1 = -2, \lambda_2 = 1$. Therefore $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} = c_1e^{-2x} + c_2e^x$.

2.2 Roots are Equal

Consider if $\lambda_{1,2} = -\frac{a}{2}$. So $a^2 - 4b = 0$ meaning that $y'' + ay' + \frac{a^2}{4}y = 0$. We know that one root is a solution so $y_1 = e^{-\frac{a}{2}x}$. Let's try $y_2 = xe^{-\frac{a}{2}x}$. Then $y_2' = e^{-\frac{a}{2}x} + \frac{-a}{2}xe^{-\frac{a}{2}x}$, $y_2'' = -ae^{-\frac{a}{2}x} + \frac{-a^2}{4}xe^{-\frac{a}{2}x}$. It can be shown that this is a solution.

2.3 Complex Roots

Complex roots are always conjugate. Also recall $e^{ix} = \cos x + i \sin x$.

Let $\lambda_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b}) = \alpha \pm i\omega$ where $\alpha = -\frac{a}{2}, \omega = \frac{\sqrt{4b-a^2}}{2}$.

$$\begin{aligned} y &= c_1' y_1 + c_2' y_2 \\ &= c_1' e^{(\alpha + j\omega)x} + c_2' e^{(\alpha - j\omega)x} \\ &= c_1' e^{\alpha x} e^{j\omega x} + c_2' e^{\alpha x} e^{-j\omega x} \end{aligned}$$

We can rewrite this as $y = (c_1' + c_2')e^{\alpha x} \cos \omega x + j(c_1' + c_2')e^{\alpha x} \sin \omega x$.

Let $c_1' = \frac{c_1 + jc_2}{2}, c_2' = \frac{c_1 - jc_2}{2}$. Since $e^{j\omega x} = \cos \omega x + i \sin \omega x$ and $e^{-j\omega x} = \cos \omega x - i \sin \omega x$.

$$y = c_1 e^{\alpha x} \cos \omega x + c_2 e^{\alpha x} \sin \omega x$$

Example 2.2 $y'' + 2y' + 2y = 0$

So $\lambda = -1 \pm j$, $\alpha = -1$, $\omega = 1$

$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$

3 Higher Order ODEs

This method will work for higher order ODEs with constant coefficients: $y^{[n]} + a_{n-1} y^{[n-1]} + \ldots a_1 y' + a_0 y = 0$.

1. Find the characteristic equation ($y = e^{\lambda x}$).

$$\lambda^{+} a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$

2. Solve for the roots (ie. using numerical solver)

3. The solution is a sum of **independent** solutions of the form

   (a) Real roots (not repeated)

   $$y_i = e^{\lambda_i x}$$

   (b) Complex roots (always in conjugate pairs)

   $$\begin{aligned} y_{i,i+1} &= e^{(\alpha \pm i\omega)x} \\ &= e^{\alpha x}(c_1 \sin \omega x + c_1 \cos \omega x) \end{aligned}$$

   (c) Real repeated roots of order m

   $$y_{i,i+1,\ldots,i+m-1} = (c_1 + c_2 x + \ldots + c_m x^{m-1})e^{\lambda x}$$

   (d) Repeated complex roots of order m

   $$y_{i,i+1,\ldots,i+2m-1} = e^{\alpha x}(c_1 \sin \omega x + c_2 \cos \omega x)(d_1 + d_2 x + \ldots d_m x^{m-1})$$

Example 3.1 $y^{[6]} - 4y^{[5]} + 14y^{[4]} + 32y^{[3]} - 79y'' + 260y' + 676y = 0$

The roots are $\lambda = -2, -2, 2 \pm 3i, 2 \pm 3i$.

$y = (c_1 + c_2 x)e^{-2x} + e^{2x}(c_3 \sin 3x + c_4 \cos 3x)(c_5 + c_6 x)$

3.1 Stability

We consister a system to be **stable** if the solution $y(t)$ to the ODE is bounded for any initial condition. That is, there exists a constant M such that $|y(t)| \leq M, \forall t \geq 0$. We consider the system to be unstable if the solution is bounded.

We can look at the roots of the characteristic polynomial to check if the solution is bounded. We need for all roots, $Re(\lambda) < 0$ or $Re(\lambda) = 0$ if not repeated.

Example 3.2 Is $y^{[6]} - 4y^{[5]} + 14y^{[4]} + 32y^{[3]} - 79y'' + 260y' + 676y = 0$ stable, with roots $\lambda = -2, -2, 2 \pm 3i$?

It is not stable due to $2 \pm 3i$ root.

3.2 Nonhomogeneous Case

Recall the method of undetermined coefficients.

Example 3.3 $y'' + 4y = 8x^2$

Look at the homogeneous case $y'' + 4y = 0$. The characteristic equation is $\lambda^2 + 4 = 0$ so $\lambda = \pm 2i$. Then the homogeneous solution is $y_h = c_1 \sin 2x + c_2 \cos 2x$.

So we try $y_p = k_2x^2 + k_1x + k_0$. Then $y_p' = 2k_2x + k_1$ and $y_p'' = 2k_2$. Substituting into the differential equation, $2k_2 + 4(k_2x^2 + k_1x + k_0) = 8x^2$. Solving yields $k_0 = -1, k_1 = 0, k_2 = 2$. Then $y = c_1 \sin 2x + c_2 \cos 2x + 2x^2 - 1$.

Example 3.4 $y'' - 3y' + 2y = e^x$

Look at homogeneous case. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$ so $\lambda_{1,2} = 1, 2$. Then $y_h = c_1e^x + c_2e^{2x}$.

We can't use $y_p = ke^x$ since it's in the equation so we'll try $y_p = kxe^x$, $y_p' = k(e^x + xe^x)$, $y_p'' = k(2e^x + xe^x)$. Substituting back, we get $k = -1$. So $y = c_1e^x + c_2e^{2x} - xe^x$.

Example 3.5 $y'' - 2y' + y = e^x + x, y(0) = 1, y'(0) = 0$

Homogeneous equation is $\lambda^2 - 2\lambda + 1 = 0$. So the homogeneous solution is $y_h = c_1e^x + c_2xe^x$. Now we look for the particular solution. Usually we try $y_p = k_1e^x + k_2x + k_3$, but k_1e^x and k_1xe^x is part of the homogeneous solution so we try $y_p = k_1x^2e^x + k_2x + k_3$ instead.

Solving for constants, we get $y_p = \frac{1}{2}x^2e^x + x + 2$. So $y = c_1e^x + c_2xe^x + x + 2 + \frac{1}{2}x^2e^x$. We must solve the intial value problem using the **particular solution**.

4 Harmonic Oscillators

Differential equation of a spring is $m\ddot{x} + c\dot{x} + kx = F$ and for an electrical system, it is $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}$. These are both in the form

$$y'' + 2\xi\omega_n y' + \omega_n^2 y = f(t)\omega_n^2$$

where ω_n = natural frequency and ξ = damping ratio. Notice that both are non-negative.

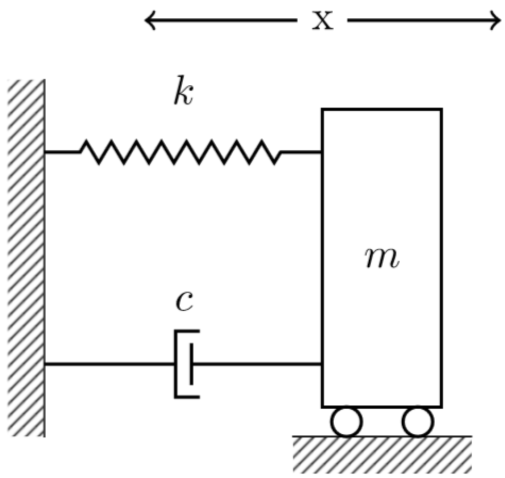


Figure 1: Mass, spring, dampener system

4.1 Homogeneous Case

Then the characteristic equation for the homogeneous case is $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$. So $\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$. We also have $\xi = \frac{c}{2\sqrt{mk}}$ and $\omega_n = \sqrt{\frac{k}{m}}$.

The real part is always negative since $\sqrt{\xi^2 - 1} < \xi$. So our solution is a negative exponential, oscillating if there are complex roots. This makes sense physically.

4.1.1 Case 1, Overdamped

In this case, $\xi > 1$. We get two real negative roots.

$$y_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

We call this overdamped.

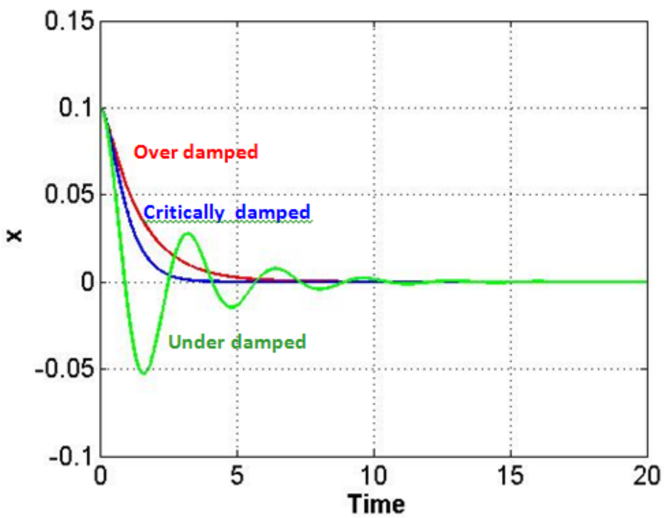


Figure 2: Overdamping, critical damping and underdamping

4.1.2 Case 2, Critical Damping

This is when $\xi = 1$. We then have repeated roots $\lambda_{1,2} = -\xi\omega_n$.

$$y_h = c_1 e^{-\xi\omega_n t} + c_2 t e^{-\xi\omega_n t}$$

This approaches 0 the fastest and is called critical damping.

4.1.3 Case 3, Underdamped

We have $\xi < 1$. Then we have two complex roots with a negative real part: $\lambda_{1,2} = -\xi\omega_n \pm \omega_n i\sqrt{1-\xi}$.

$$y_h = e^{-\xi\omega_n t}(A\cos(\omega t) + B\sin(\omega t))$$

where $\omega = \omega_n\sqrt{1-\xi^2}$

4.2 Nonhomogeneous Case

Equations are in the form $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = f(t)\omega_n^2$. We need to look at the three cases: overdamped, critically damped and underdamped. All those cases were stable for the homogeneous case.

Suppose $f(t) = F_n\cos(\omega t)$. Using the method of undetermined coefficients, the form of the solution is $y(t) = y_h(t) + A\cos\omega t + B\sin\omega t$. Substituting and simplifying, we get

$$y = y_h + \underbrace{\underbrace{\frac{(\omega_n^2 - \omega^2)(\omega_n^2 F_0)}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}}_{u}\cos\omega t + \underbrace{\frac{2\xi\omega_n\omega(\omega_n^2 F_0)}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega\omega_n)^2}}_{v}\sin\omega t}_{y_p}$$

We want to transform the sin and cos into just one cos in the form $y_p = E\cos(\omega t + \Phi)$.

$$\begin{aligned} y_p &= E\cos(\omega t + \Phi) \\ &= E\cos\Phi\cos\omega t - E\sin\Phi\sin\omega t \end{aligned}$$

Then $u = E\cos\Phi$ and $v = E\sin\Phi$ so $E = \sqrt{u^2 + v^2}$ and $\Phi = \tan^{-1}\frac{-v}{u}$.

The particular solution is oscillating at the same frequency as the input but with different amplitudes and a phase shift. Now consider when there is no damping:

$$y = A\cos\omega_n t + B\sin\omega_n t + \frac{\omega_n^2 F_0}{\omega_n^2 - \omega^2}\cos\omega t$$

4.2.1 Beating

Suppose ω_n is close to ω and $y(0) = 0$ and $\dot{y}(0) = 0$. Then we get

$$y(t) = \underbrace{2\frac{F_0\omega_n^2}{\omega_n^2 - \omega^2}\sin(\frac{\omega_n - \omega}{2}t)}_{\text{envelope}}\ \underbrace{\sin(\frac{\omega_n + \omega}{2}t)}_{\text{same frequency as } f(t)}$$

4.2.2 Resonance

If $\omega_n = \omega$ then we can't use this equation since the forcing function is the same form as y_h. We have to try $y_p = t(A\cos\omega_n t + B\sin\omega_n t)$. And that blows up linearly.

5 Laplace Transform

5.1 Definition

Let $f(t)$ be defined for all $t \geq 0$. Then

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

is called the Laplace transform of $f(t)$ for all s such that this improper integral exists. Also s is a complex number.

The inverse is denoted as $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

5.2 Table of Transforms

$f(t), t \geq 0$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$

5.3 Derivation of Transforms

Proof 5.1 Consider the Heaviside function. For $t \geq 0$, $f(t) = 1$.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}(-1)dt \\
&= \lim_{T\to\infty} \left(-\frac{1}{s}e^{-st}\right)\Big|_0^T \\
&= \lim_{T\to\infty} \left[-\frac{1}{s}e^{-st} - \frac{-1}{s}\right] \\
&= \frac{1}{s} && \text{(if Re}(s) > 0 \text{ then this converges)}
\end{aligned}$$

Then for the Heaviside function, $F(s) = \frac{1}{s}$. The region of convergence is $\mathrm{Re}(s) > 0$ because otherwise the limit is not defined.

Proof 5.2 Consider $f(t) = e^{at}, t \geq 0$.

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at} dt \\ &= \lim_{T\to\infty} \int_0^T e^{-(s-a)t} dt \\ &= \lim_{T\to\infty} \frac{1}{a-s} e^{-(s-a)t}\Big|_0^T \end{aligned}$$

If $\text{Re}(s-a) > 0$, then as $T \to \infty$ the first term goes to zero. Then

$$F(s) = \frac{1}{s-a}$$

for $\text{Re}(s) > a$.

We can keep doing this for other functions to generate tables.

The Laplace transform is **linear** (ie. $\mathcal{L}\{af(t) + bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$)

Proof 5.3

$$\begin{aligned} \mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty e^{-st}(af(t) + bg(t))dt \\ &= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt \\ &= aF(s) + bG(s) \end{aligned}$$

Note that the region of convergence is the intersection of the region of convergence of the original functions.

The inverse Laplace transform is also linear.

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\,\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\}$$

Example 5.1 $f(t) = \cosh at$

$$\begin{aligned} F(s) &= \mathcal{L}\{\cosh at\} \\ &= \mathcal{L}\{\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}\} \\ &= \frac{1}{2}\,\mathcal{L}\{e^{at}\} + \frac{1}{2}\,\mathcal{L}\{e^{-at}\} \\ &= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} \\ &= \frac{s}{s^2 - a^2} \end{aligned}$$

The region of convergence is $\text{Re}(s) > a$ and $\text{Re}(s) > -a$. So we want $\text{Re}(s) > a$.

Example 5.2 $F(s) = \frac{1}{(s-1)(s-2)}$.

$$\begin{aligned}
F(s) &= \mathcal{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
&= e^{2t} - e^{t}
\end{aligned}$$

Proof 5.4

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

We showed earlier that this holds for $n = 0$ with the step function.

Assume for some n that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

$$\begin{aligned}
\mathcal{L}\{t^{n+1}\} &= \int_0^\infty e^{-st} t^{n+1} dt && \text{(integrate by parts: } v = t^{n+1}, du = e^{-st}dt) \\
&= \underbrace{-\frac{1}{s} e^{-st} t^{n+1}\Big|_0^\infty}_{0 \text{ if } \mathrm{Re}(s)>0} + \frac{n+1}{s} \underbrace{\int_0^\infty e^{-st} t^n dt}_{\mathcal{L}\{t^n\}} \\
&= 0 + \frac{n+1}{s} \frac{n!}{s^{n+1}} \\
&= \frac{(n+1)!}{s^{n+2}}
\end{aligned}$$

So the formula is true by induction. $\square$

Proof 5.5

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Remember that s is in the complex domain.

$$\begin{aligned}
\mathcal{L}\{\sin \omega t\} &= \mathcal{L}\left\{\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\} \\
&= \frac{1}{2i}\left(\mathcal{L}\{e^{i\omega t}\} - \mathcal{L}\{e^{-i\omega t}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s - i\omega}\frac{s + i\omega}{s + i\omega} - \frac{1}{s + i\omega}\frac{s - i\omega}{s - i\omega}\right) \\
&= \frac{1}{2i}\left(\frac{s + i\omega}{s^2 + \omega^2} - \frac{s - i\omega}{s + \omega^2}\right) \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$

$\square$

5.4 Existance of Laplace Transform

Not all $f(t)$ have a Laplace transform. We need the integrand $e^{-st}f(t)$ to go to zero sufficiently fast.

For example, $e^{-st}t^n, e^{-st}e^{\omega t}$ can be made to decay sufficiently fast. But $e^{-st}e^{t^2}$ will not converge for any s.

$f(t)$ does not need to be continuous. Assume $f(t)$ is piecewise continuous. That is, it is continuous on all but a finite number of points and at those points, it has finite left and right-sided limits (only finite jumps).

Theorem 5.1 *Let $f(t)$ be a function that is piecewise continuous on every finite interval for $t > 0$ and $|f(t)| \leq Me^{at}$ for some a. Then the Laplace transform exists for* $\mathrm{Re}\{s\} > a$ *and is unique except at the discontinuities.*

Example 5.3 $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$. $\frac{1}{\sqrt{t}}$ is not continuous at $t = 0$ but is continuous everywhere else for $t > 0$ so the Laplace transform exists.

$\mathcal{L}\left\{e^{t^2}\right\}$ does not exist since there does not exist a such that $|f(t)| \leq Me^{at}$.

Example 5.4 What is the Laplace transform of

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt \\
&= \int_0^1 e^{-st}tdt && \text{(since } f(t) = 0 \text{ for } t \geq 0\text{)} \\
&= -\frac{1}{s}e^{-st}t\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt && \text{(integration by parts)} \\
&= -\frac{1}{se^s} + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) \\
&= -\frac{1}{se^s} - \frac{1}{s^2e^s} + \frac{1}{s^2} \\
&= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)
\end{aligned}$$

5.5 Inverse Laplace

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{st}ds$$

This is an integration in the complex plane. The α depends on the region of convergence. If $F(s)$ is rational, we can use tables and partial fraction expansions.

Example 5.5

$$\begin{aligned} F(s) &= \frac{3s^2+s-7}{s^3-7s^2} \\ &= \frac{3}{s-7} + \frac{1}{s^2} \\ f(t) &= 3e^{7t} + t, t \geq 0 \end{aligned}$$

5.6 Application to ODEs

We need to figure out what happens when we differentiate a function.

Theorem 5.2 *Suppose $f(t)$ is continuous for $t \geq 0$ and satisfies the conditions to have a Laplace transform. As well, suppose $\frac{df(t)}{dt}$ is piecewise continuous on every finite interval in $t \geq 0$. Then, there exists an α such that $\mathcal{L}\left\{\frac{df(t)}{dt}\right\}$ exists for $\mathrm{Re}\{s\} > \alpha$ and*

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

where $f(0)$ is the initial value.

Proof 5.6

$$\begin{aligned} \mathcal{L}\left\{\frac{df}{dt}\right\} &= \int_0^\infty e^{-st}f'(t)dt \\ &= \underbrace{e^{-st}}_{u}\underbrace{f(t)}_{v}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\,\mathcal{L}\{f(t)\} \end{aligned}$$

$\square$

Although not shown, we should keep track of the region of convergence.

We can apply this twice to get

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= s\,\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - f'(0) \\
&= s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0)
\end{aligned}$$

Or generally,

$$\mathcal{L}\left\{\frac{df^n(t)}{dt^n}\right\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{[n-1]}(0)$$

We can now solve ODEs with initial values using algebraic techniques.

Example 5.6 $\dot{x} = x, x(0) = 1$

$$\begin{aligned}
\mathcal{L}\{\dot{x}\} &= \mathcal{L}\{x\} \\
sX(s) - x(0) &= X(s) \\
X(s) &= \frac{1}{s - x(0)} \\
X(s) &= \frac{1}{s-1} \\
x(t) &= e^t, t \geq 0 && \text{(taking the inverse Laplace)}
\end{aligned}$$

Example 5.7 $\ddot{y} = 1, y(0) = 0, \dot{y}(0) = 1$

$$\begin{aligned}
\mathcal{L}\{\ddot{y}\} &= \mathcal{L}\{1\} \\
s^2Y(s) - sy(0) - y'(0) &= \frac{1}{s} \\
s^2Y(s) &= \frac{1}{s} + 1 \\
Y(s) &= \frac{1}{s^2} + \frac{1}{s^3} \\
y(t) &= t + \frac{t^2}{2}, t \geq 0
\end{aligned}$$

Example 5.8 Find $\mathcal{L}\{\cos\omega t\}$.

If $f = \sin\omega t,\ f(0) = 0, f' = \omega\cos\omega t$.

$$
\begin{aligned}
\mathcal{L}\{f'\} &= s\,\mathcal{L}\{f\} - f(0) \\
\mathcal{L}\{\omega\cos\omega t\} &= s\frac{\omega}{s^2+\omega^2} - 0 \\
\mathcal{L}\{\cos\omega t\} &= \frac{s}{s^2+\omega^2}
\end{aligned}
$$

Example 5.9 $\ddot{y} + 9y = 1, y(0) = 0, \dot{y}(0) = 1$

$$
\begin{aligned}
\mathcal{L}\{\ddot{y} + 9y\} &= \mathcal{L}\{1\} \\
s^2\,\mathcal{L}\{y\} - sy(0) - \dot{y}(0) + 9\,\mathcal{L}\{y\} &= \frac{1}{s} \\
\mathcal{L}\{y\}(s^2+9) &= \frac{1}{s} + sy(0) + \dot{y}(0) \\
\mathcal{L}\{y\} &= \frac{\frac{1}{s}+1}{s^2+9} \\
\mathcal{L}\{y\} &= \frac{1}{s(s^2+9)} + \frac{1}{s^2+9} \\
\mathcal{L}\{y\} &= \frac{1}{9s} - \frac{1}{9}\frac{s}{s^2+3^2} + \frac{1}{3}\frac{3}{s^2+3^2} \\
\mathcal{L}\{y\} &= \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{1}{3}\sin 3t
\end{aligned}
$$

5.7 Shifting Theorems and the Heaviside Function

5.7.1 s-Shifting

Theorem 5.3 *If $f(t)$ has a transform $F(s)$ where $s > \alpha$, then $e^{at}f(t)$ has the transform $F(s-a)$ where $s - a > \alpha$.*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Proof 5.7

$$
\begin{aligned}
F(s-a) &= \int_0^\infty e^{-(s-a)}f(t)dt \\
&= \int_0^\infty e^{-st}(e^{at}f(t))dt \\
&= \mathcal{L}\{e^{at}f(t)\}
\end{aligned}
$$

□

We can now use partial fraction expansion of all rational functions in s using our previous table.

$f(t), t \geq 0$	$F(s)$
$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Example 5.10

$$
\begin{aligned}
F(s) &= \frac{1}{s^2(s^2+4s+40)} \\
&= \frac{1}{s^2\left((s+2)^2+6^2\right)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \underbrace{\frac{C(s+2)}{(s+2)^2+6^2} + \frac{6D}{(s+2)^2+6^2}}_{\text{form where we can directly take inverse}} \\
f(t) &= \mathcal{L}^{-1}\left\{-\frac{1}{400} + \frac{1}{40s^2} + \frac{1}{400}\frac{s+2}{(s+2)^2+6^2} - \frac{1}{300}\frac{6}{(s+2)^2+6^2}\right\} \\
&= -\frac{1}{400} + \frac{1}{40}t + \frac{1}{400}e^{-2t}\cos 6t - \frac{1}{300}e^{-2t}\sin 6t
\end{aligned}
$$

Or solve using complex domain

Example 5.11

$$
\begin{aligned}
F(s) &= \frac{1}{s^2(s^2+4s+40)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2+6i} + \frac{D}{s+2-6i} \\
f(t) &= A + Bt + Ce^{(-2-6i)t} + De^{(-2+6i)t}
\end{aligned}
$$

C and D will be a complex conjugate pair.

Example 5.12 $\ddot{y} + 2\dot{y} + 5y = 0, y(0) = 2, \dot{y}(0) = -4$

Taking Laplace,

$$
\begin{aligned}
s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 5Y(s) &= 0 \\
Y(s) &= \frac{2s}{s^2 + 2x + 5} \\
&= \frac{2s}{(s+1)^2 + 2^2} \\
&= \frac{A(s+1)}{(s+1)^2 + 2^2} + \frac{2B}{(s+1)^2 + 2^2}
\end{aligned}
$$

Solving gives $A = 2, B = -1$.

$$
\begin{aligned}
Y(s) &= \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2} \\
y(t) &= 2e^{-t}\cos 2t - e^{-t}\sin 2t
\end{aligned}
$$

Example 5.13 $\ddot{y} - 2\dot{y} + y = e^t + t, y(0) = 1, \dot{y}(0) = 0$

$$
\begin{aligned}
\frac{1}{s-1} + \frac{1}{s^2} &= s^2Y(s) - sy(0) - y'(0) - 2(sY(s) + y(0)) + Y(s) \\
Y(s) &= \frac{s-2}{(s-1)^2} + \underbrace{\frac{1}{(s-1)^3}}_{\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{t^2}{2}} + \frac{1}{s^2(s-1)^2} \\
&= \left(\frac{1}{s-1} - \frac{1}{(s-1)^2}\right) + \frac{e^t t^2}{2} + \left(\frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1}{s^2} + 2\frac{1}{s}\right) \\
&= e^t - te^t + \frac{e^t t^2}{2} + te^t - 2e^t + t + 2 \\
&= -e^t + t + 2 + \frac{t^2 e^t}{2}
\end{aligned}
$$

Example 5.14 $\ddot{y} + 6\dot{y} + 13y = 1, y(0) = 0, \dot{y}(0) = 0$

$$
\begin{aligned}
s^2Y(s) - sy(0) - \dot{y}(0) + 6(sY(s) - y(0)) + 13Y(s) &= \frac{1}{s} \\
(s^2 + 6s + 13)Y(s) &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s^2 + 6s + 13)} \\
Y(s) &= \frac{1}{s((s+3)^2 + 4)} \\
Y(s) &= \frac{A}{s} + \frac{B(s+3)}{(s+3)^2 + 2^2} + \frac{2C}{(s+3)^2 + 2^2} \\
Y(s) &= A + Be^{-3t}\cos 2t + Ce^{-3t}\sin 2t
\end{aligned}
$$

5.7.2 Time Shifting

Define the Heaviside Step Function $u(t-a)$ as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

Theorem 5.4 *A delayed function*

$$\widetilde{f}(t) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

has Laplace transform $e^{-as}\mathcal{L}\{f(t)\}$.

Equivalently,

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Proof 5.8

$$\begin{aligned} e^{-as}F(s) &= e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau \\ &= \int_0^\infty e^{-s(\tau+a)}f(\tau)d\tau \\ &= \int_a^\infty e^{-st}f(t-a)dt && \text{(with } t=\tau+a\text{)} \\ &= \int_0^\infty e^{-st}u(t-a)f(t-a)dt \end{aligned}$$

Example 5.15 Show $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

Since $\mathcal{L}\{u(t)\} = \frac{1}{s}$ then $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$.

Example 5.16 Find $\mathcal{L}^{-1}\{\frac{e^{-3s}}{s^3}\}$.

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{1}{s^3}\} &= \frac{t^2}{2} \\ \mathcal{L}^{-1}\{\frac{e^{-3s}}{s^3}\} &= \frac{(t-3)^2}{2}u(t-3) \end{aligned}$$

Example 5.17 Find the transform of

$$f(t) = \begin{cases} 2 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \\ \sin t & t \geq 2\pi \end{cases}$$

We'll write $f(t)$ in terms of Heaviside.

From 0 to π, $f(t) = 2u(t)$
From 0 to 2π, $f(t) = 2u(t) - 2u(t-\pi)$

For all t,

$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\underbrace{\sin(t-2\pi)}_{\mathcal{L}\{\sin t\}=\frac{1}{s^2+1}}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - e^{-\pi s}\frac{2}{s} + e^{-2\pi s}\frac{1}{s^2+1}$$

Notice that we have to shift $\sin t$ by 2π to get it into a form where we can apply the theorem.

Example 5.18

$$\begin{aligned} F(s) &= \frac{2}{s^2} - 2\frac{e^{-2s}}{s^2} - 4\frac{e^{-2s}}{s} + s\frac{e^{-\pi s}}{s^2-1} \\ f(t) &= 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi) \end{aligned}$$

Example 5.19 $\ddot{y} - 4\dot{y} + 4y = t + 2u(t-3), y(0) = 0, \dot{y}(0) = 0$

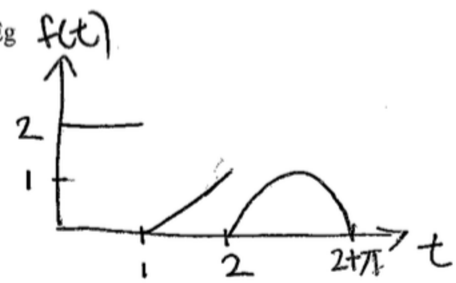
$$\begin{aligned} \ddot{y} - 4\dot{y} + 4y &= t + 2u(t-3) \\ s^2Y(s) - sy(0) - \dot{y}(0) - 4(sY(s) - y(0)) + 4Y(s) &= \frac{1}{s^2} + 2\frac{e^{-3s}}{s} \\ (s^2 - 4s + 4)Y(s) &= \frac{1}{s^2} + 2\frac{e^{-3s}}{s} \\ (s-2)^2Y(s) &= \frac{1}{s^2} + 2\frac{e^{-3s}}{s} \\ Y(s) &= \frac{1}{s^2(s-2)^2} + \frac{2e^{-3s}}{s(s-2)^2} \end{aligned}$$

After partial fraction expansion,

$$\begin{aligned} Y(s) &= \left(\frac{1}{4s} + \frac{1}{4s^2} - \frac{1}{4(s-2)} + \frac{1}{4(s-2)^2}\right) + e^{-3s}\left(\frac{1}{2s} - \frac{1}{2(s-2)} + \frac{1}{(s-2)^2}\right) \\ &= \frac{1}{4} + \frac{t}{4} - \frac{e^{2t}}{4} + \frac{te^{2t}}{4} + \frac{u(t-3)}{2} - \frac{e^{2(t-3)}u(t-3)}{2} + (t-3)e^{2(t-3)}u(t-3) \end{aligned}$$

Example 5.20

Eg



$$\begin{aligned} f(t) &= 2 - 2u(t-2) + (t-1)u(t-1) - (t-2)u(t-2) - 1u(t-2) + \sin(t-2)u(t-2) \\ &= \frac{2}{s} - \frac{2e^{-s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2+1} \end{aligned}$$

Example 5.21 $\mathcal{L}\{u(t-1)t\}$

$$\begin{aligned} f(t) &= tu(t-1) \\ &= (t-1)u(t-1) + u(t-1) \\ &= \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} \end{aligned}$$

5.8 Periodic Signals

Theorem 5.5 *If f is periodic with period T and piece-wise continuous over this period, then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \underbrace{\int_0^T f(t)e^{-st}dt}_{\text{Laplace of one period}}$$

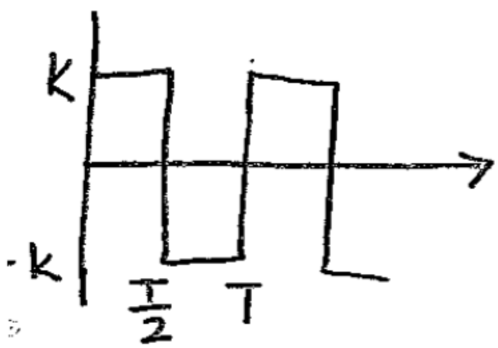
Periodic means $f(t) = f(t+T)$.

Note: if we take the Laplace transform over one period, we must make sure that the function is equal to 0 for $t > T$.

Proof 5.9

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^T e^{-st} f(t)dt + \underbrace{\int_T^{2T} e^{-st} f(t)dt}_{t=\tau+T} + \underbrace{\int_{2T}^{3T} e^{-st} f(t)dt}_{t=\tau+2T} + ... \\
&= \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(\tau+T)} \underbrace{f(\tau+T)}_{f(\tau)} d\tau + \int_0^T e^{-s(\tau+2T)} \underbrace{f(\tau+2T)}_{f(\tau)} d\tau + ... \\
&= \int_0^T e^{-st} f(t)dt + e^{-st} \int_0^T e^{-st} f(t)dt + e^{-2st} \int_0^T e^{-st} f(t)dt + ... \\
&= (1 + e^{-st} + e^{-2st} + ...) \int_0^T e^{-st} f(t)dt \\
&= \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)dt \qquad \text{(geometric series with } |e^{-st}| < 1)
\end{aligned}
$$

Example 5.22



Looking at one period,

$$f_T(t) = k - 2k\ u(t - \frac{T}{2}) + k\ u(t - T)$$

Then the Laplace transform for one period is

$$\mathcal{L}\{f_T(t)\} = \frac{k}{s} - \frac{2k}{s} e^{-\frac{T}{w}s} + \frac{k}{s} e^{-Ts^2}$$

Using the previous theorem,

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{sT}} \left(\frac{k}{s} - \frac{2k}{s} e^{-\frac{T}{2}s} + \frac{k}{s} e^{-Ts} \right)$$

5.9 Integration of f(t)

Theorem 5.6 *If $f(t)$ is piecewise-continuous and satisfies $|f(t)| \leq Me^{\lambda t}$, then*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\,\mathcal{L}\{f(t)\}$$

Proof 5.10 One can show that if $|f(t)| \leq Me^{\lambda t}$ then $\int_0^t f(\tau)d\tau$ will have a Laplace transform.

Let $g(t) = \int_0^t f(\tau)d\tau$ then $f(t) = \frac{dg(t)}{dt}$.

Now one can show that $|g(t)| \leq \frac{Me^{\lambda t}}{\lambda}$ which means there is a Laplace transform.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{g'(t)\} \\
\mathcal{L}\{f(t)\} &= s\,\mathcal{L}\{g(t)\} - \cancelto{0}{g(0)} \\
\mathcal{L}\{g(t)\} &= \frac{1}{s}\,\mathcal{L}\{f(t)\}
\end{aligned}$$

5.10 Control Systems

Assume our system initially has zero initial conditions. The forcing function $u(t)$ is considered the input to the system and the output is the variable $y(t)$ that we are trying to solve for.

$$\underbrace{a_0 y^{[n]}(t) + ...a_n y(t)}_{\text{outputs}} = \underbrace{b_0 u^{[n-1]}(t) + ... + b_{n-1}u(t)}_{\text{inputs}}$$

Taking derivatives with zero initial conditions are essentially multiplication by s. Then taking the Laplace transform gives

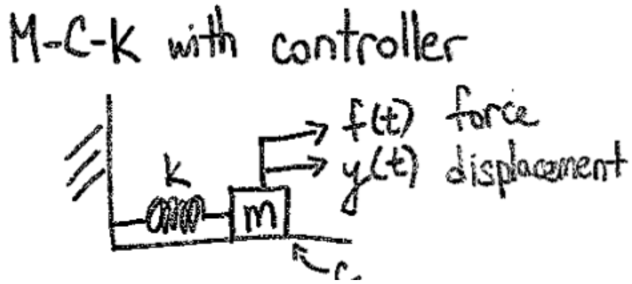
$$\begin{gathered}
Y(s)(a_0 s^n + ... + a_n) = U(s)(b_0 s^{n-1} + ... + b_{n-1}) \\
Y(s) = G(s)U(s)
\end{gathered}$$

where

$$G(s) = \frac{b_0 s^{n-1} + ... + b_{n-2}s + b_{n-1}}{a_0 s^n + ... + a_{n-1}s + a_n}$$

$G(s)$ is called the **transfer function**.

This is useful for block diagrams, for example mass spring damper system with a controller. We want to change the damping characteristics to achieve critical damping.



The equations are $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 f(t)$. Or in the Laplace domain,

$$Y(s) = \underbrace{\left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right)}_{G(s)} F(s)$$

The force is typically created by a motor. A computer takes signals from sensors and outputs a signal which drives the motor.

We can represent the computer as a transfer function and connect it to our system above, called the **plant** $G(s)$ to modify its behavior.

The most common way to control the system is to use a PD (proportional derivative) controller where we make the controller behave as the following ODE:

$$f(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

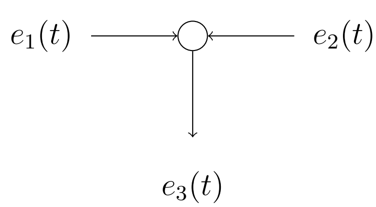
Or in the Laplace domain

$$F(s) = \underbrace{(K_p + sK_d)}_{C(s)} E(s)$$

Note that we can't truly do a derivative of a real signal but we can get a good approximation.

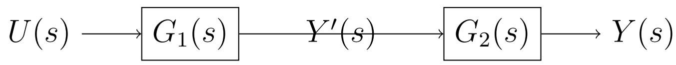
Basic block operations:

1. Summing node ($e_3(t) = e_1(t) + e_2(t)$)

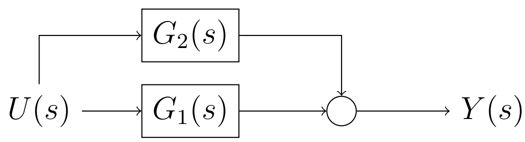
   By linearity, $E_3(s) = E_1(s) + E_2(s)$.

2. Series



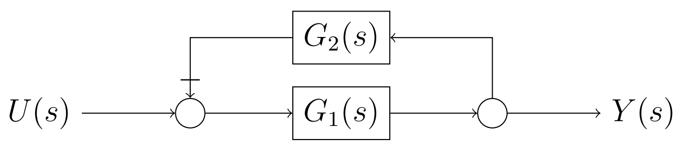
$Y(s) = G_2(s)G_1(s)U(s)$

3. Parallel



$Y(s) = (G_1(s) + G_2(s))U(s)$

4. Feedback (negative)

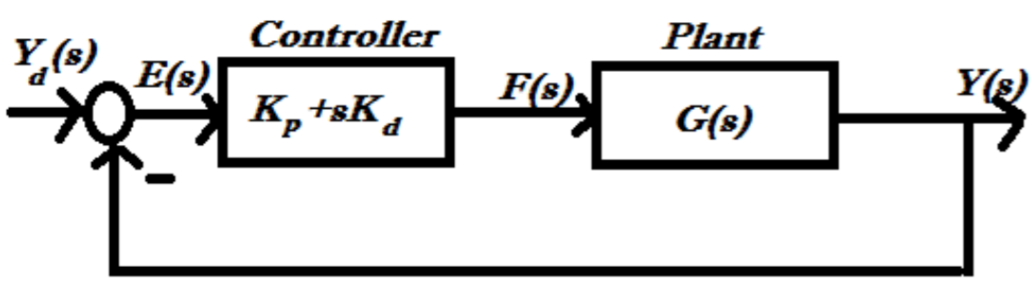


$$
\begin{aligned}
Y(s) &= G_1(s)e(s) \\
Y(s) &= G_1(s)(U(s) - G_2(s)Y(s)) && \text{(since } E(s) = U(s) - G_2(s)Y(s)\text{)} \\
Y(s) &= \frac{G_1(s)}{1 + G_1(s)G_2(s)}U(s)
\end{aligned}
$$

Let $e(t) = y_d(t) - y(t)$ where $e(t)$ is an error signal and $y_d(t)$ is the desired value of the signal in question. If $e(t)$ approaches zero, then $y(t)$ approaches $y_d(t)$. In the laplace domain this is

$$E(s) = Y_d(s) - Y(s)$$

This gives the following block diagram



This is called closed loop feedback. The controller $C(s)$ used is called a PD controller.

Using the block operations above, the overall transfer function is

$$Y(s) = \frac{(K_p + sK_d)G(s)}{1 + (K_p + sK_d)G(s)} Y_d(s)$$
$$Y(s) = \frac{\omega_n^2(K_p + K_d s}{s^2 + (2\xi\omega_n + K_d\omega_n^2)s + (K_p\omega_n^2 + \omega_n^2)} Y_d(s)$$

Now changing K_p and K_d allows us to change the roots of the denominator. This allows us to modify the behavior of the system.

If $Y(s) = G(s)Y_d(s)$ is some general transfer function, then we can multiply out everything to get the RHS of the equation to be a ratio of polynomials in s. When we do the PFE, all the terms in the RHS will be terms of the roots of the denominator polynomial and those of the input $Y_d(s)$.

A transfer function is **Bounded Input, Bounded Output stable (BIBO)** if for ALL bounded input, the output is always bounded.

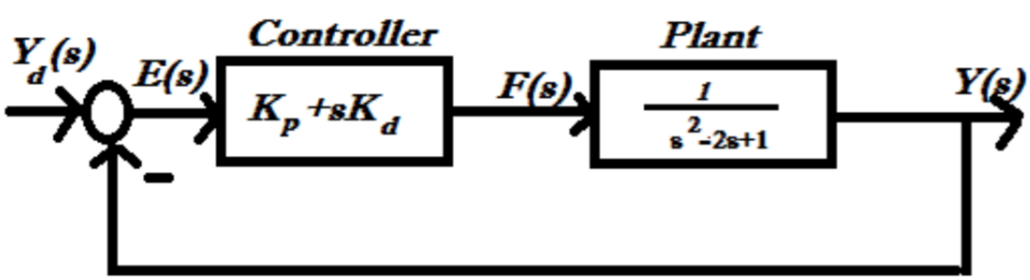
The **poles** of the transfer function are the roots of the denominator polynomial.

The **zeroes of the transfer function** are the roots of the numerator polynomial. This is not important for stability.

Theorem 5.7 *The transfer function is BIBO stable if and only if all the poles of the transfer function have a negative real part.*

Note that poles with zero real part may be unstable for some inputs due to repeated roots.

Example 5.23 Suppose we have the following system. Design a conroller to make this system stable, critically damped and decay as fast as e^{-2t} and te^{-2t}.



Notice that the plant is unstable with poles at $-1, -1$.

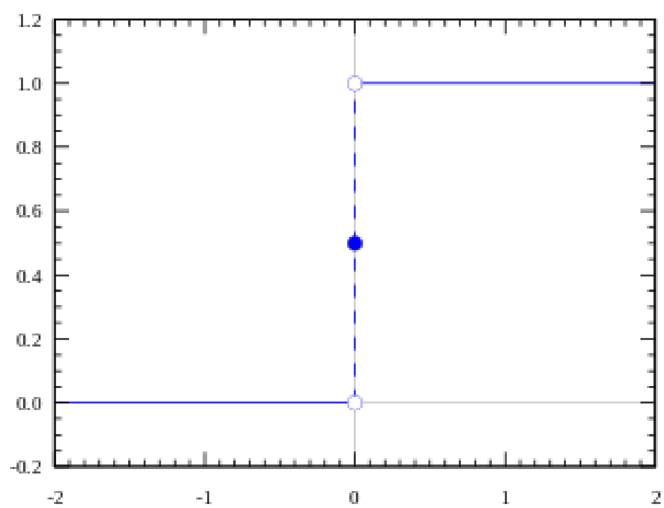
$$
\begin{aligned}
Y(s) &= \frac{C(s)G(s)}{1+C(s)G(s)}Y_d(s) \\
&= \left[\frac{(K_p+sK_d)\frac{1}{s^2-2s+1}}{1+(K_p+sK_d)\frac{1}{s^2-2s+1}}\right]Y_d(s) \\
&= \left[\frac{K_p+sK_d}{s^2-2s+1+K_p+sK_d}\right]Y_d(s) \\
&= \frac{K_p+sK_d}{s^2+(K_d-2)s+(1+K_p)}Y_d(s)
\end{aligned}
$$

We want the poles at $-2,-2$. Then we want the denominator to look like $(s+2)(s+2) = s^2+4s+4$. Then by comparing coefficients, $K_p = 3, K_d = 6$.

Recall, $Y(s) = G(s)U(s)$. If $U(s) = 1$, then $Y(s) = G(s)$ or $y(t) = g(t)$.

What input has Laplace transform of 1?

Motivation: $1 = \frac{1}{s}s$ (ie. derivative of step function)



We can approximate this as a function

$$
g_{\Delta T,a}(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{\Delta T} & a \le t < a+\Delta T \\ 1 & t \ge a+\Delta T \end{cases}
$$

Then as $\Delta T \to 0, a \to 0$, we approach a step function.

We'll consider the derivative to this approximation $f(t) = g'(t)$.

$$f(t)_{\Delta T,a}(t) = \begin{cases} 0 & t < a \\ \frac{1}{\Delta T} & a \leq t \leq a + \Delta T \\ 0 & t > a + \Delta T \end{cases}$$

$$\begin{aligned} f_{\Delta T,a}(t) &= \frac{1}{\Delta T}u(t-a) - \frac{1}{\Delta T}u(t-a-\Delta T) \\ F_{\Delta T,a}(s) &= \frac{e^{-at}}{s\Delta T} - \frac{e^{-(a+\Delta T)s}}{s\Delta T} \\ F_{\Delta T,a}(s) &= e^{-at}\frac{1-e^{-(\Delta T)s}}{s\Delta T} \end{aligned}$$

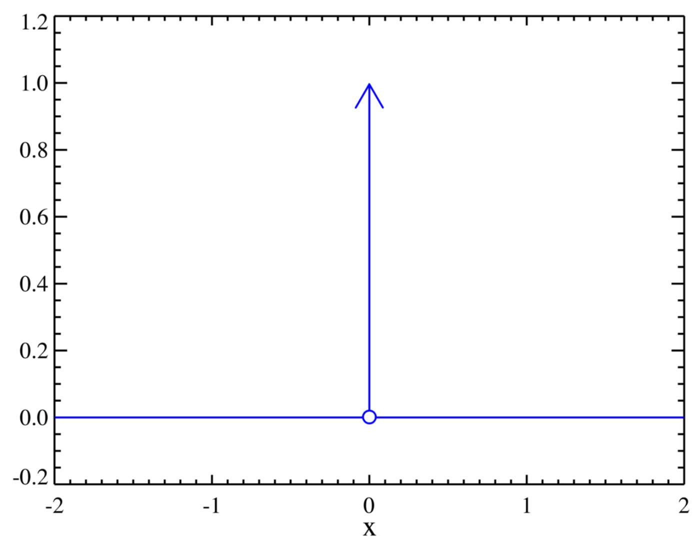
Consider the limit as $\Delta T \to 0$,

$$\begin{aligned} \lim_{\Delta T \to 0} F_{\Delta T,a}(s) &= \lim_{\Delta T \to 0} e^{-as}\frac{\frac{\partial}{\partial \Delta T}(1-e^{-\Delta T s})}{\frac{\partial}{\partial \Delta T}s} && \text{(Using L'Hopital's Rule)} \\ &= \lim_{\Delta T \to 0} e^{-as}\frac{se^{-\Delta T s}}{s} \\ &= e^{-as} \end{aligned}$$

We define $\delta(t-a) \equiv \lim_{\Delta T \to 0} f_{\Delta T,a}(t)$. This is called the **Dirac Delta function**. Note that the Dirac delta function is not an actual function since it has a non-zero integral with only one point of discontinuity. The integral of this signal is called an impulse.

We have $\mathcal{L}\{\delta(t-a)\} = \lim_{\Delta T \to 0} F_{\Delta T,a}(s) = e^{-as}$. In particular, $\mathcal{L}\{\delta(t)\} = 1$.

We represent this graphically as



Properties:

1.
$$\int_0^\infty \delta(t-a)dt = 1$$

2. Filtering property:
$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

Proof 5.11
$$\int_0^\infty f(t)\delta(t-a)dt = f(a)$$

By the Mean Value Theorem, there exists t_0 such that $f(t_0) = \int_a^{a+\Delta T} f(t)dt$.

$$\begin{aligned}
\int_0^\infty f(t)\delta(t-a)dt &= \lim_{\Delta T\to 0}\int_0^\infty f(t)\frac{1}{\Delta T}[u(t-a)-u(t-(a+\Delta T))]dt \\
&= \lim_{\Delta T\to 0}\frac{1}{\Delta T}\int_a^{a+\Delta T} f(t)dt \\
&= \lim_{\Delta T\to 0}\frac{1}{\Delta T}f(t_0)\Delta T && \text{(Mean Value Theorem)} \\
&= f(a) && \text{(as } t_0 \to a \text{ when } \Delta T \to 0\text{)}
\end{aligned}$$

□

Now recall $Y(s) = G(s)U(s)$. If $U(s) = 1$ (in other words, an impulse), then $Y(s) = G(s)$ or taking the inverse Laplace transform, $y(t) = g(t)$ is called the impulse response.

If there are zero initial conditions and $u(t) = \delta(t-a)$, then the resulting response is the inverse Laplace transform of $G(s)e^{-as}$ or the impulse response time shifted by a to give $g(t-a)H(t-a)$ where we use $H(t-a)$ for the Heaviside function instead of u to avoid confusion with the input.

In practice, we often hit a linear time invariant system with an impulse to check this impulse response. We need to normalize so that the integral is 1. This gives us $g(t)$.

Recall $Y(s) = G(s)U(s)$. Can we find $y(t)$ from $g(t)$ for any arbitrary $u(t)$? We need the concept of **convolution**.

5.11 Convolution

Theorem 5.8 *(Convolution Theorem)*

Let $u(t)$ and $g(t)$ satisfy the conditions that guarantee the existence of the Laplace transforms. Then, the product of their transforms $Y(s) = G(s)U(s)$ is the transform of the ***convolution*** *of $g(t)$ and $u(t)$ which is*

$$y(t) = g(t) * u(t) = \int_0^t u(\tau)g(t-\tau)d\tau$$

Proof 5.12 Our system is linear therefore superposition holds. If we have a $u(t)$ made of Diract delta functions, we can sum up time shifted $g(t)$ to get the response.

Let $f_{\Delta T}(t - n\Delta T)$ generate an impulse response at $n\Delta T$.

For any arbitrary input,

$$\begin{aligned}
y(t) &\approx \sum_{n=0}^{\infty} u(n\Delta T) f_{\Delta t}(t - n\Delta T)\Delta T \\
y(t) &= \sum_{n=0}^{\infty} u(n\Delta T)\Delta T[g(t - n\Delta T)H(t - n\Delta T)] \\
y(t) &= \sum_{n=0}^{n'} u(n\Delta T) g(t - n\Delta T)\Delta T \qquad\qquad (H(t - n\Delta T) = 0 \text{ when } n\Delta T > t)
\end{aligned}$$

where n' is the smallest n such that $n\Delta T > t$.

As $\Delta T \to 0$, we get a Riemann integral

$$y(t) = \int_0^t u(\tau) g(t - \tau) d\tau$$

Proof 5.13

$$\begin{aligned}
F(s)G(s) &= \left(\int_0^\infty e^{-st} f(\tau) d\tau\right)\left(\int_0^\infty e^{-su} g(u) du\right) \\
&= \int_0^\infty \left(\int_\tau^\infty e^{-s(\tau+u)} f(\tau) g(u) du\right) d\tau
\end{aligned}$$

Substituting $t = \tau + u$ and noting that τ is fixed in the interior integral so $du = dt$,

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau) g(t - \tau) dt d\tau \\
&= \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t - \tau) H(t - \tau) dt d\tau \\
&= \int_0^\infty \int_0^\infty e^{-st} f(\tau) g(t - \tau) H(t - \tau) d\tau dt \\
&= \int_0^\infty \left(\int_0^t e^{-st} f(\tau) g(t - \tau) d\tau\right) dt \\
&= \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t - \tau) d\tau\right) dt \\
F(s)G(s) &= \mathcal{L}\left\{\int_0^t f(\tau) g(t - \tau) d\tau\right\}
\end{aligned}$$

5.11.1 Properties

1. Commutative
$$f * g = g * f$$
2. Distributive
$$f * (g + h) = f * g + f * h$$
3. Associative
$$(f * g) * h = f * (g * h)$$
4. Zero Element
$$0 * f = f * 0 = 0$$
5. Identity
$$f * \delta = \delta * f = f$$

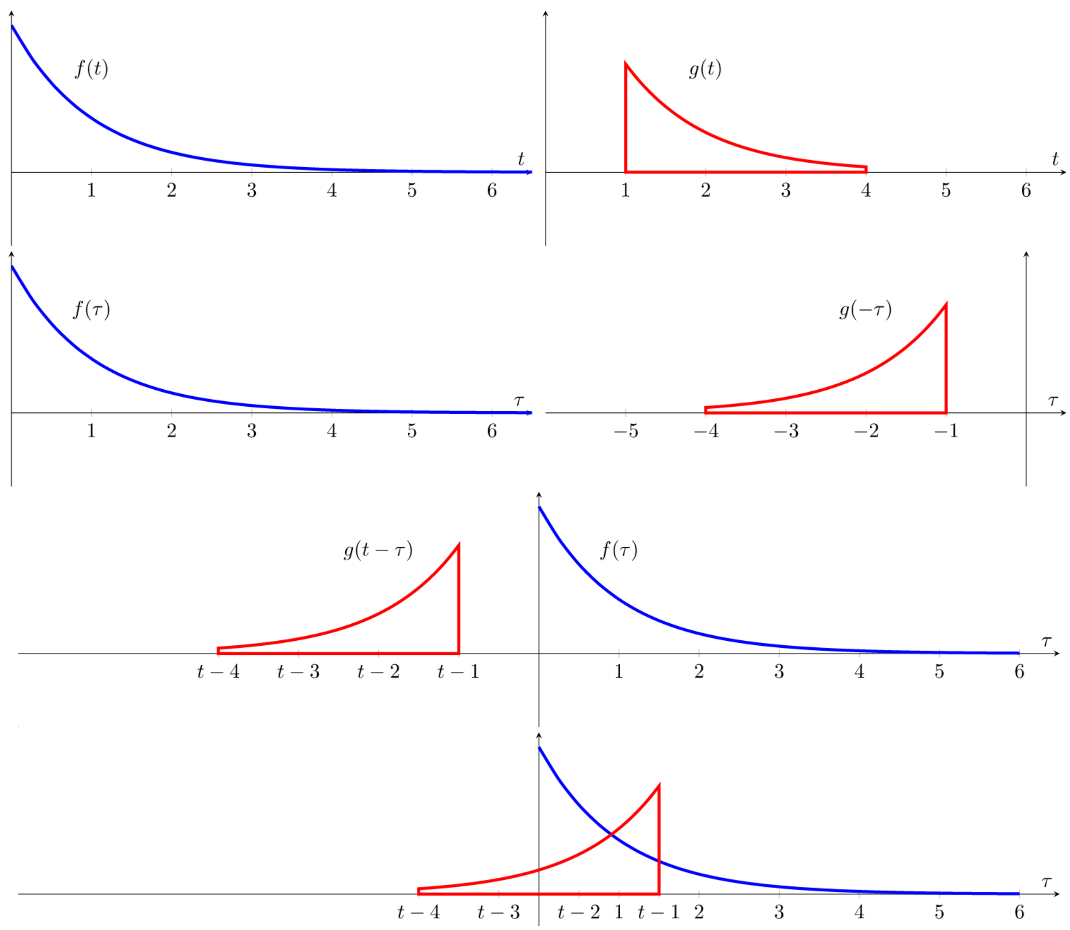
Proof 5.14 Proofs for all of these can be done in the Laplace domain using the convolution theorem.

We'll show $f * (g + h) = f * g + f * h$.

$$\begin{aligned} \mathcal{L}\{f * (g + h)\} &= \mathcal{L}\{f\}\,\mathcal{L}\{f + h\} \\ &= \mathcal{L}\{f\}\,\mathcal{L}\{g\} + \mathcal{L}\{f\}\,\mathcal{L}\{h\} \\ f * (g + h) &= f * g + f * h \end{aligned}$$

□

The convolution integral can be visualized as reflecting one function and taking the weighted sum as the function is translated from $-\infty$ to ∞.



5.12 Simultaneous Differential Equations

There are times when two or more ODEs are coupled together. Suppose we have a circuit with the vollowing ODEs

$$\frac{di_1}{dt} + \frac{di_2}{dt} + 56i_1 + 40i_2 = 400$$
$$\frac{di_2}{dt} - 8i_1 + 10i_2 = 0$$

We can use Laplace transforms and since we solve using algebra, we have a system of n equations and n unknown

$$(s+56)I_1(s) + (s+40)I_2(s) = \frac{400}{s}$$
$$-8I_1(s) + (s+10)I_2(s) = 0$$

Solving using algebra,

$$I_2(s) = \frac{3200}{s(s+59.1)(s+14.9)}$$
$$i_2(t) = 3.64 + 1.22e^{-59.1t} - 4.86e^{-14.9t}$$

Similarly

$$i_1(t) = 4.55 - 7.49e^{-59.1t} + 2.98e^{-14.9t}$$

6 Fourier Series

6.1 Useful Concepts

If a function is periodic with period p

$$f(t) = f(t+p)$$

A function periodic with period p is also periodic with period $2p, 3p$, etc. The smallest such period p is called the **fundamental period**. Note that this exntends from $-\infty$ to ∞.

An even function is symmetric about $x = 0$ such that $f(-x) = f(x)$.

An odd function is antisymmetric about $x = 0$ such that $f(-x) = -f(x)$.

Properties

1. even + even = even
2. even $\times$ even = even
3. odd + odd = odd
4. odd $\times$ odd = even
5. even $\times$ odd = odd

Theorem 6.1

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & f(x) \textit{ is even} \\ 0 & f(x) \textit{ is odd} \end{cases}$$

The following functions all have period $2L$

$$1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, ..., \cos\frac{m\pi x}{L}, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, ..., \sin\frac{m\pi x}{L}$$

Then the following also has period $2L$

$$a_0+a_1\cos\frac{\pi x}{L}+a_2\cos\frac{2\pi x}{L}+...+a_m\cos\frac{m\pi x}{L}+...+b_1\sin\frac{\pi x}{L}+b_2\sin\frac{2\pi x}{L}+...+b_m\sin\frac{m\pi x}{L}+...$$
$$=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right]$$

If this converges, this will be periodic with period $2L$. We'll ignore convergence since proofs are extremely involved.

Theorem 6.2 *(Orthogonality Properties)*

Analagous to vector dot products

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=0 \quad n\neq m$$
$$\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=0 \quad n\neq m$$
$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=0 \quad \forall n,m$$

We can use the following properties to prove the Orthogonality properties

$$\sin x\sin y=\frac{1}{2}(-\cos(x+y)+\cos(x-y))$$
$$\cos x\cos y=\frac{1}{2}(\cos(x+y)+\cos(x-y))$$
$$\sin x\cos y=\frac{1}{2}(\sin(x+y)+\sin(x-y))$$

Proof 6.1

$$\begin{aligned}\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx&=\int_{-L}^{L}\frac{1}{2}\left(\cos\frac{n\pi x+m\pi x}{L}+\cos\frac{n\pi x-m\pi x}{L}\right)dx\\&=-\frac{1}{2}\left(\frac{L}{n\pi+m\pi}\sin\frac{(n+m)\pi x}{L}+\frac{L}{n\pi-m\pi}\sin\frac{(n-m)\pi x}{L}\right)\Big|_{-L}^{L}\\&=0 \qquad \text{(since } \sin n\pi=0 \text{ for all integers } n)\end{aligned}$$

□

Proof 6.2

$$\begin{aligned}\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx&=\int_{-L}^{L}\frac{1}{2}\left(-\cos\frac{n\pi x+m\pi x}{L}+\cos\frac{n\pi x-m\pi x}{L}\right)dx\\&=\frac{1}{2}\left(\frac{L}{n\pi+m\pi}\sin\frac{(n+m)\pi x}{L}-\frac{L}{n\pi-m\pi}\sin\frac{(n-m)\pi x}{L}\right)\Big|_{-L}^{L}\\&=0 \qquad \text{(since } \sin n\pi=0 \text{ for all integers } n)\end{aligned}$$

□

Proof 6.3

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\left(\sin\frac{n\pi x + m\pi x}{L} + \sin\frac{n\pi x - m\pi x}{L}\right)dx \\
&= 0 && \text{(since sin is an odd function)}
\end{aligned}$$

□

Theorem 6.3

$$\int_{-L}^{L}\cos^2\frac{n\pi x}{L} = L$$
$$\int_{-L}^{L}\sin^2\frac{n\pi x}{L} = L$$

Proof 6.4

$$\begin{aligned}
\int_{-L}^{L}\cos^2\frac{n\pi x}{L} &= \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{2n\pi x}{L} + 1\right)dx \\
&= \frac{1}{2}\left[\frac{L}{2n\pi}\sin\frac{2n\pi x}{L} + x\right]\Big|_{-L}^{L} \\
&= L
\end{aligned}$$

□

6.2 Fourier Series

If the function $f(x)$ we are trying to represent is periodic,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$$

This is the **Fourier series**. We will assume that the equality holds and that we can interchange an integration and an infinite summation (not necessarily the case).

We can find coefficients using integration. To find a_m, multiply by $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\begin{aligned}
\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \underbrace{\int_{-L}^{L} a_0\cos\frac{m\pi x}{L}dx}_{\text{0 by orthogonality when } n=0} + \sum_{n=1}^{\infty}\left[a_n\underbrace{\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}}_{\text{0 except for n=m}} + b_n\underbrace{\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}}_{\text{0 by orthogonality}}\right] \\
&= La_m \\
a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx
\end{aligned}$$

For a_0, simply integrate from $-L$ to L (we are multiplying by $\cos 0 = 1$).

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \left(a_0 + \sum_{n=1}^{\infty} \left[a_n \underbrace{\cos \frac{n\pi x}{L}}_{\text{0 by orthogonality when } n=0} + b_n \underbrace{\sin \frac{n\pi x}{L}}_{\text{odd function}} \right] \right)$$
$$= 2La_0$$
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

Note that this is the average value of the function. You can often "eyeball" this.

For b_m we similarly multiply by $\sin \frac{m\pi x}{L}$ and integrate both sides from $-L$ to L

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx = \int_{-L}^{L} a_0 \sin \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \left[a_n \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \right]$$
$$= b_m L$$
$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx$$

The coefficients are called the Fourier coefficients and the resulting summation is the Fourier Series.

Theorem 6.4 *Let f be $2L$ periodic and let f and f' be piecewise continuous on the interval from $-L$ to L. Then the Fourier Series converges to $f(x)$ at every point of x where f is continuous and to the mean value $\frac{f(x^+)+f(x^-)}{2}$ where discontinuous.*

Recall, $f(x)$ is piecewise continuous if it has right and left hand limits that are finite.

Example 6.1 Consider the following function, periodic with period 2π

$$f(x) = \begin{cases} -k & -\pi < x \leq 0 \\ k & 0 < x \leq \pi \end{cases}$$

The period is 2π so $L = \pi$. We'll solve for the coefficients

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = 0$$

since it is an odd function (or by eyeballing the average value).

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{L}dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos(nx)dx + \int_{0}^{\pi} k\cos(nx)dx\right] \\
&= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] \\
&= 0
\end{aligned}
$$

Or we can notice that $f(x)$ is odd and $\cos\frac{n\pi x}{L}$ is even so the integral of the product is 0. Now solve for b_n,

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin(nx)dx + \int_{0}^{\pi} k\sin(nx)dx\right] \\
&= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right] \\
&= \frac{2k}{n\pi}\left(1 - \frac{\cos n\pi}{2} - \frac{\cos -n\pi}{2}\right) \\
&= \frac{2k}{n\pi}(1 - \cos n\pi)
\end{aligned}
$$

$cosn\pi$ is 1 when n is even and -1 when n is odd. Then

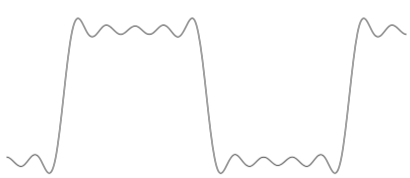
$$
b_n = \begin{cases} \frac{4k}{n\pi} & \text{n is odd} \\ 0 & \text{n is even} \end{cases}
$$

So then we have

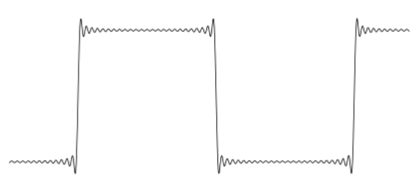
$$
\begin{aligned}
f(x) &= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right) \\
&= \frac{4k}{\pi}\sum_{\sigma=0}^{\infty}\frac{\sin(2\sigma+1)x}{2\sigma+1}
\end{aligned}
$$

Gibbs Phenomenon

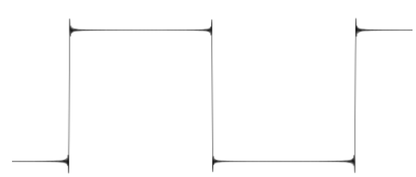
Looking at the discontinuity as we add more harmonics.



Using 5 harmonics



Using 25 harmonics



Using 125 harmonics

Notice that the overshoot narrows but does not go down. We always have an overshoot.

Theorem 6.5 *A Fourier series of an even function of period $2L$ is a Fourier Cosine series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

and

$$a_0 = \frac{1}{L} \int_0^L f(x)dx$$
$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Proof 6.5 $\cos \frac{n\pi x}{L}$ is even and $\sin \frac{n\pi x}{L}$ is odd. So $f(x)\cos \frac{n\pi x}{L}$ is even and $f(x)\sin \frac{n\pi x}{L}$ is odd.

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)dx \\ &= \frac{1}{L} \int_0^L f(x)dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \\ &= 0 \end{aligned}$$

□

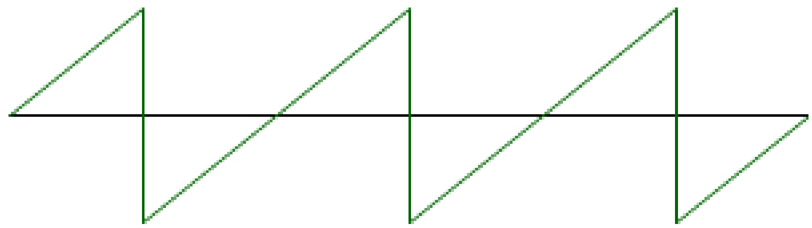
Theorem 6.6 *A Fourier series of an odd function of period $2L$ is a Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

and the coefficients are

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Example 6.2 The Sawtooth function



Let $f(x) = x, -\pi < x \leq \pi$ and be periodic with period 2π. $f(x)$ is odd so $a_0 = 0$ and $a_n = 0$. Now we'll solve for b_n

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \; dx \\ &= \frac{2}{\pi} \left(\frac{\sin nx - nx \cos nx}{n^2} \right) \Big|_0^{\pi} \\ &= -\frac{2}{n} \cos nx \\ &= -\frac{2}{n} (-1)^n \end{aligned}$$

Then

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$

Example 6.3 Consider the previous function shifted by 0.5 to the right. That is, $f(x) = x - 0.5, -\pi + 0.5 < x \leq \pi + 0.5$.

We can let $x' = x - 0.5$ and find the Fourier series with respect to x'. Then substitute $x' = x - 0.5$ to get the Fourier series with respect to x.

And for all $f(x)$ periodic with period $2L$, *any* interval $(x_0, x_0 + 2L)$ can be used.

6.3 Complex Fourier Series

We can write rewrite our Fourier series as

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\left[a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right]\\
&= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L}+\frac{b_n}{2}\sin\frac{n\pi x}{L}\right]+\sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L}+\frac{b_n}{2}\sin\frac{n\pi x}{L}\right]\\
&= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L}+\frac{b_n}{2}\sin\frac{n\pi x}{L}\right]+\sum_{n=0}^{\infty}\left[\frac{a_{-n}}{2}\cos\frac{-n\pi x}{L}+\frac{b_{-n}}{2}\sin\frac{-n\pi x}{L}\right]\\
&= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L}+\frac{b_n}{2}\sin\frac{n\pi x}{L}\right]+\sum_{n=0}^{-\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L}+\frac{b_n}{2}\sin\frac{n\pi x}{L}\right]
\end{aligned}$$

where $a_n = a_{-n}$ and $b_n = -b_{-n}$ since cos is even and sin is odd.

Then we have

$$f(x) = \sum_{n=-\infty}^{\infty}\left[a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right]$$

where

$$a_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \text{ and } b_n = \frac{1}{2L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Now we also have

$$\cos\frac{n\pi x}{L} = \frac{e^{\frac{in\pi x}{L}}+e^{\frac{-in\pi x}{L}}}{2} \text{ and } \sin\frac{n\pi x}{L} = \frac{e^{\frac{in\pi x}{L}}-e^{\frac{-in\pi x}{L}}}{2i}$$

Then

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty}\left[a_n\frac{e^{\frac{in\pi x}{L}}+e^{\frac{-in\pi x}{L}}}{2}+b_n\frac{e^{\frac{in\pi x}{L}}-e^{\frac{-in\pi x}{L}}}{2i}\right]\\
&= \sum_{n=-\infty}^{\infty}\left[\frac{a_n-ib_n}{2}e^{\frac{in\pi x}{L}}\right]+\sum_{n=-\infty}^{\infty}\left[\frac{a_n+ib_n}{2}e^{\frac{-in\pi x}{L}}\right]\\
&= \sum_{n=-\infty}^{\infty}\left(\frac{a_n-ib_n}{2}+\frac{a_{-n}+ib_{-n}}{2}\right)e^{\frac{in\pi x}{L}}\\
&= \sum_{n=-\infty}^{\infty}(a_n-ib_n)e^{\frac{in\pi x}{L}} && \text{(since } a_n=a_{-n}, b_n=-b_n\text{)}
\end{aligned}$$

If we let $c_n = a_n - ib_n$, we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

$$
\begin{aligned}
c_n &= a_n - ib_n \\
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx - \int_{-L}^{L} f(x) i \sin\frac{n\pi x}{L} \\
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\frac{-in\pi x}{L}} dx
\end{aligned}
$$

Note that c_n and c_{-n} are complex conjugates. Also e^{inx} and e^{-inx} are complex conjugates. Therefore the $+|n|$ and $-|n|$ terms are complex conjugates.

Example 6.4 Find the Fourier series for a periodic function f defined by $f(x) = e^x$ on $-\pi < x < \pi$.

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-\frac{in\pi x}{\pi}} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)} dx \\
&= \frac{1}{2\pi}\frac{1}{1-in} e^{(1-in)x}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{1}{1-in}(e^{-in\pi}e^{\pi} - e^{in\pi}e^{-\pi})
\end{aligned}
$$

Notice that $e^{in\pi} = e^{-in\pi} = (-1)^n$

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\frac{1}{1-in}(-1)^n(\underbrace{e^{\pi} - e^{-\pi}}_{2\sinh\pi}) \\
&= \frac{\sinh}{\pi}(-1)^n \frac{1+in}{1+n^2}
\end{aligned}
$$

Then

$$
f(x) = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{1+in}{1+n^2}e^{inx}
$$

6.4 Amplitude Spectrum

The Fourier series is helpful in finding how much signal there is at each frequency.

The plots are done using real or complex representation, but usually the complex.

$$
f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}
$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx$$

and let $\omega_0 = \frac{\pi}{L}$ (the fundamental frequency).

It turns out that the power at each frequency is given by $|c_0|^2$ for the constant component and $|c_{-n}|^2 + |c_n|^2 = 2|c_n|^2$ for the nth component. This is Parseval's Theorem.

$$P = \frac{1}{T}\int_{C}^{C+T} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

6.5 Fourier Integral

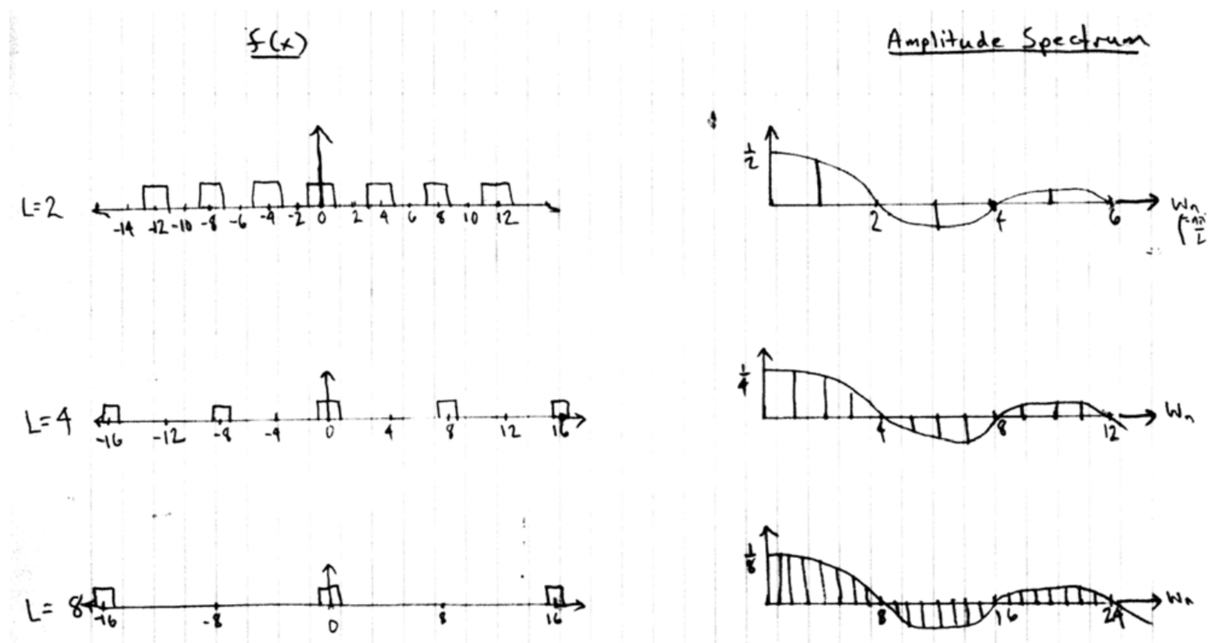
Now if the signal is aperiodic. For example,

$$f(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 \le x < 1 \\ 0 & 1 \le x \le L \end{cases}$$

Then we have a Fourier Cosine Series

$$a_0 = \frac{1}{L}, b_n = 0, a_n = \frac{2\sin\frac{n\pi}{L}}{n\pi}$$

What happens as $L \to \infty$? If we let $\omega_n = \frac{n\pi}{L}$ then $\omega_n \to 0$. So we have more points in the amplitude spectrum that are more densely packed. In other words, we approach a continuum.



Now,

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos(\omega_n x)\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv\right]$$

Let $\Delta\omega = \omega_{n+1} - \omega_n$. Then

$$\begin{aligned}\frac{(n+1)\pi}{L} - \frac{n\pi}{L} &= \frac{\pi}{L}\\ \frac{1}{L} &= \frac{\Delta\omega}{\pi}\end{aligned}$$

Assume $\int_{-\infty}^{\infty}|f(x)|dx$ is finite (absolutely integrable). Then the first term approaches 0 as L approaches ∞.

$$f_L = \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos(\omega_n x)\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv\right]\Delta\omega$$

This is a Reimann sum so we have

$$f_L = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos(\omega x)\underbrace{\int_{-L}^{L} f(v)\cos(\omega v)dv}_{A(\omega)} + \sin\omega_n x\underbrace{\int_{-L}^{L} f(v)\sin(\omega v)dv}_{B(\omega)}\right]d\omega$$

Theorem 6.7 *If $f(x)$ is piecewise continuous in every finite interval and has a right hand and left hand derivative at every point, and if $\int_{-\infty}^{\infty}|f(x)|dx < \infty$, then $f(x)$ can be represented by*

$$f_L(x) = \int_{0}^{\infty}[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv$.

At each point where $f(x)$ is discontinuous, then $f(x) = \frac{f(x_-)+f(x_+)}{2}$.

Note that the theorem does not require that the function be periodic.

When finding the coefficients, the same principle applies in recognizing even and odd functions. That is, odd functions only involve $B(\omega)$ and even functions only involve $A(\omega)$.

The Gibbs phenomenon also still holds at discontinuities. The peak narrows as $L \to \infty$.

Example 6.5 Consider the function

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise}\end{cases}$$

This is an even function so $B(\omega) = 0$. Now consider $A(\omega)$

$$\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv \\
&= \frac{2}{\pi}\int_{0}^{\infty} \underbrace{f(v)\cos(\omega v)}_{\text{even}} dv \\
&= \frac{2}{\pi}\int_{0}^{1} \cos(\omega v)dv \\
&= \frac{2\sin(\omega v)}{\pi\omega}\Big|_0^1 \\
&= \frac{2\sin\omega}{\pi\omega}
\end{aligned}$$

Then

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos(\omega x)\frac{\sin\omega}{\omega}d\omega$$

Note that at $x = 1, -1$ the Fourier integral converges to $\frac{1}{2}$.

For this example, instead of integrating to infinity, let's integrate to L. The approximation is

$$f_L(x) = \frac{2}{\pi}\int_0^L \cos\omega x\frac{\sin\omega}{\omega}d\omega$$

As $L \to \infty$, we get closer and closer to the square function. This is the Gibbs phenomenon. It also happens at discontinuities for Fourier integrals.

Now, we'll substitute the complex exponential for the sin and cos terms

$$\begin{aligned}
f(x) &= \int_0^\infty \left[A(\omega)\frac{e^{i\omega x} + e^{-i\omega x}}{2} + B(\omega)\frac{e^{i\omega x} - e^{-i\omega x}}{2i}\right] d\omega \\
&= \int_0^\infty \left[\underbrace{\frac{A(\omega) - iB(\omega)}{2}}_{C(\omega)} e^{i\omega x} + \underbrace{\frac{A(\omega) + iB(\omega)}{2}}_{\bar{C}(\omega)} e^{-i\omega x}\right] d\omega \\
&= \int_0^\infty C(\omega)e^{i\omega x}d\omega + \int_0^\infty \bar{C}(\omega)e^{-i\omega x}d\omega && (\text{let } \omega' = \omega) \\
&= \int_0^\infty C(\omega)e^{i\omega x}d\omega + \int_0^{-\infty} \bar{C}(-\omega')e^{i\omega' x}(-d\omega') && (\bar{C}(-\omega) = C(\omega)) \\
f(x) &= \int_{-\infty}^\infty C(\omega)e^{i\omega x}d\omega
\end{aligned}$$

where

$$C(\omega) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} f(v)\cos\omega v\ dv - i\int_{-\infty}^{\infty} f(v)\sin\omega v\ dv\right]$$

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\underbrace{(\cos\omega v - i\sin\omega v)}_{e^{-i\omega v}}dv$$

Therefore

$$f(x) = \int_{-\infty}^{\infty} C(\omega)e^{i\omega x}d\omega \text{ where } C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv$$

Example 6.6

$$f(x) = xe^{-|x|}$$

The function is continuous and odd. Is it absolutely integrable ($\int_{\infty}^{\infty}|f(x)|dx$ is bounded)?

$$\begin{aligned}\int_{-\infty}^{\infty}|f(x)|dx &= \int -\infty^{\infty}\underbrace{|x|e^{-|x|}}_{\text{even}}dx\\ &= 2\int_0^{\infty} xe^{-x}dx\\ &= 2\end{aligned}$$

Then $f(x)$ is bounded so we can have a complex Fourier integral. Now we solve for $C(\omega)$.

$$\begin{aligned}C(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} te^{-|t|}e^{-i\omega t}dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{0} te^{t}e^{-i\omega t}dt + \frac{1}{2\pi}\int_0^{\infty} te^{-t}e^{-i\omega t}dt\\ &= -\frac{2i\omega}{(i+\omega^2)\pi}\end{aligned}$$

Then

$$xe^{-|x|} = \frac{-2i}{\pi}\int_{-\infty}^{\infty}\frac{\omega}{(1+\omega^2)^2}e^{i\omega x}d\omega$$

6.6 Fourier Transform

This leads to the **Fourier Transform** with a reshuffling of constants.

Given that a function f is piecewise continuous on $[-L, L]$ for any L.

Suppose $\int_{-\infty}^{\infty} |f(t)|dt$ ($f(t)|$ is absolutely integrable). The Fourier transform of f is

$$F[f(t)] = \hat{f}(w) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

And the inverse Fourier Transform is

$$f(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}dw = F^{-1}[\hat{f}(w)]$$

The Fourier transform gives us an indication of the "amount of signal" at any frequency w. The amplitude spectrum is a graph of $|F(w)|$ vs w.

The aboslutely integrable conditions rules out a number of functions (eg. x, 2, $e^a t$, $\sin \omega t$). In practice, we look at finite signals with a start and end. Piecewise continuous finite signals are always absolutely integrable.

Example 6.7 Find the Fourier transform of $f(t) = e^{-at}$, $a > 0, t > 0$.

Note that this is absolutely integrable, but we won't show it here.

$$\begin{aligned} F[f(t)] &= \int_{-\infty}^{\infty} e^{-at}H(t)e^{-i\omega t}dt \\ &= \int_{0}^{\infty} e^{-(a+i\omega)t}dt \\ &= \frac{1}{a+i\omega} \end{aligned}$$

6.6.1 Properties

1. Linearity
$$F[af(t) + bg(t)] = aF[f(t)] + bF[g(t)]$$

2. Time-shifting
$$F[f(t-t_0)] = e^{-i\omega t_0}\hat{f}(\omega)$$

3. Frequency-shift property
$$F[e^{i\omega_0 t}f(t)] = \hat{f}(\omega - \omega_0)$$

4. Differentiation Property

   If $f(x)$ is continuous and $\hat{f}(x) \to 0$ as $|x| \to \infty$ and $f'(x)$ is absolutely integrable, then
$$F[f'(x)] = i\omega\hat{f}(\omega)$$

5. Convolution Property
$$F[f(x) * g(x)] = \hat{f}(\omega)\hat{g}(\omega)$$
6. Parseval's Identity
$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$
This means we can look at power either in time or frequency.

6.6.2 Relationship with Laplace Transform

For Laplace, if $f(t) = 0$ for $t < 0$ then

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

For Fourier, $f(t) = 0$ for $t < 0$ and **absolutely integrable** then

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)H(t)e^{-i\omega t}dt = \int_0^{\infty} f(t)e^{-i\omega t}dt$$

Then the two are identical except $s = i\omega$.

Example 6.8 $f(t) = H(t-1)e^{-(t-1)}sin(t-1)$ for $t \geq 0$. Find $\hat{f}(\omega)$.

This will be absolutely integrable. This is bounded by e^{-t} which is absolutely integrable.

$$\mathcal{L}\{f(t)\} = e^{-s}\frac{1}{(s+1)^2+1}$$
$$\hat{f}(\omega) = e^{-i\omega}\frac{1}{(i\omega+1)^2+1}$$

7 Partial Differential Equations

These are differential equations of several variables.

The notation for partial derivates with x, t has independent variables and u as the dependent variable:

$$\frac{\partial u(x,t)}{\partial x} \equiv u_x, \frac{\partial^2 u(x,t)}{\partial x^2} \equiv u_{xx}, \frac{\partial^2 u(x,t)}{\partial x \partial t} \equiv u_{xt}$$

There usually is no set methodology to solve PDEs. Often we cannot solve exactly, but only approximately.

We will only examine the three most common PDE forms seen by engineers:

1. Laplace equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

   Describes the steady state heat equation, electrostatic potential in a uniform dielectric, steady state shape of an elastic membrane.

2. Wave Equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2}$$

   Describes propagation of electromagnetic waves, sound vibrations

3. Heat Equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{1}{k}\frac{\partial u(x,y,z,t)}{\partial t}$$

   Describes how heat is transferred from a hot area to a cold area by conduction.

The order of a PDE is the highest partial derivative appearing in the equation. For example, all the above PDEs are second order.

The Laplacian operator is defined as

$$\mathcal{L}(u) \equiv \nabla^2 u \equiv \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$$

A linear PDE is one that satisfies

$$\mathcal{L}(\alpha u + \beta v) = \alpha\,\mathcal{L}(u) + \beta\,\mathcal{L}(v)$$

where $u(x,t)$ and $v(x,t)$ are two functions. All the above examples are linear.

Example 7.1 $\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} = 0$ is non linear.

Homogeneous PDEs is where

$$\mathcal{L}\{u\} = 0$$

Non-homogeneous PDEs is where

$$\mathcal{L}\{u\} = f(u)$$

for $f(u) \neq 0$.

A linear combination of solutions to a linear homogeneous PDE is also a solution. If $u_1, ..., u_m$ are solutions to $\mathcal{L}\{u\} = 0$ then so is $\sum_{i=1}^{m} c_i u_i$ where c_i are constants.

Any solution to the nonhomogeneous PDE is called a particular solution. If $\mathcal{L}\{u_p\} = f(u_p)$ then u_p is a particular solution.

If u is some solution to the linear homogeneous ODE, then the set of all solutions to the nonhomogeneous linear PDE is $u + u_p$ for some $u \in S$ where S is the set of all solutions to the homogeneous case.

There are constant coefficient linear PDEs. There can also be systems of PDEs. We will only look at the three examples, however. We will also not look at modelling.

We often cannot find ALL the solutions of the linear homogeneous ODE. In some cases, there will be an infinite number of linearly independent solutions. We have no guarantee that these are ALL of the solutions but if they allow us to satisfy the initial conditions and boundary conditions, everything still works practically.

7.1 Classification of Second Order Linear PDEs

We'll consider the case when there are only two independent variables. Second order linear PDEs take the form of

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

where A, B, C are functions of x, y and D can be a function of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}$.

1. Parabolic if $B^2 - AC = 0$

   For example, the heat equation is parabolic. If we hold y and z constant,

   $$\alpha^2 u_{xx} = \underbrace{u_t}_{y \text{ in above}}$$

   Then $A = \alpha^2, B = 0, C = 0$ so $B^2 - AC = 0$.

2. Hyperbolic if $B^2 - AC > 0$

   For example the wave equation is hyperbolic. If we hold y and z constant,

   $$c^2 u_{xx} = u_{tt}$$

   Then $A = c^2, B = 0, C = -1$ so $B^2 - AC = c^2 > 0$.

3. Elliptic if $B^2 - AC < 0$

   For example, the Laplace Equation is elliptic. If we hold z constant,

   $$u_{xx} + u_{yy} = 0$$

   Then $A = 1, B = 0, C = 1$ so $B^2 - AC = -1 < 0$.

7.2 Wave Equation

In general, the wave eqaution is

$$c^2\nabla^2 u = u_{tt}$$

where c is the speed of propagation.

Let's consider a simpler example which is one dimensional. Let us consider a vibrating string where the speed of propagation is given by

$$c^2 u_{xx} = u_{tt}$$

As an aside, the speed of propagation is $c = \sqrt{\frac{T}{\mu}}$ where T is the tension and μ is the linear mass density from first year physics.

We will solve for u for $0 < x < \pi$. We need conditions at both these ends. These are called the **boundary conditions**. Assume the string is fixed at either end,

$$u(0,t) = 0, u(\pi,t) = 0$$

We also need **initial conditions**. Suppose the string starts at rest with the following condition.

$$u(x,0) = \begin{cases} x & 0 \leq x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \leq \pi \end{cases}$$

For exam, when solving PDEs we should state

1. Picture of System
2. Equation
3. Boundary Condition
4. Initial Conditions

A trivial solution of this is $u(x,t) = 0$ which satisfies the boundary conditions but not the initial conditions.

To solve we use **separation of variables** and let:

$$u(x,t) = X(x)T(t)$$

Then differentiating,

$$u_{xx} = X''(x)T(t)$$
$$u_t t = X(x)T''(t)$$

So substituting into our wave equation,

$$\begin{aligned} X''(x)T(t)c^2 &= X(x)T''(t) \\ \frac{X_{xx}}{X} &= \frac{T_{tt}}{c^2T} = \lambda \end{aligned}$$

Since we have a function of x equal to λ and a function of t equal to λ, λ must be a constant. This can now be written as

$$\begin{aligned} T_{tt} &= \lambda c^2 T \\ T_{tt} - \lambda c^2 T &= 0 \\ X_{xx} &= \lambda X \\ X_{xx} - \lambda X &= 0 \end{aligned}$$

Also note our boundary conditions are $u(0,t) = u(\pi,t) = 0$. Then substituting for u, $X(0)T(t) = X(\pi)T(t) = 0$ so

$$X(0) = X(\pi) = 0$$

The X equation is second order. Depending on the value of λ, there are three cases:

1. $\lambda = 0$

   From $X_{xx} - \lambda X = 0$, $X_{xx} = 0$:

   $$X(x) = Ax + B$$

   Our boundary conditions give

   $$\begin{aligned} X(0) &= 0 \\ 0A + B &= 0 \\ B &= 0X(\pi) & &= 0 \\ \pi A + B &= 0 \end{aligned}$$

   So $A = B = 0$. Then $X(x) = 0$ so $u(x,t) = 0T(t) = 0$. This is a trivial solution so we ignore it.

2. $\lambda > 0$

   We have $X_{xx} - \lambda X = 0$. Solving the characteristic equation gives roots of $\pm\sqrt{\lambda}$:

   $$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

Our boundary conditions give

$$\begin{aligned} X(0) &= 0 \\ A + B &= 0 \\ X(\pi) &= 0 \\ Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} &= 0 \end{aligned}$$

So $A = B = 0$. This is a trivial solution so we discard it.

3. $\lambda < 0$

   As before, our characteristic equation gives roots of $\pm\sqrt{\lambda} = \pm\sqrt{-\lambda}i$.

$$X(x) = A\sin(-\sqrt{-\lambda}x) + B\cos(-\sqrt{-\lambda}x)$$

The our boundary conditions give

$$\begin{aligned} X(0) &= 0 \\ 0 &= 0A + B \\ B &= 0 \\ X(\pi) &= 0 \\ A\sin(\sqrt{-\lambda}\pi) &= 0 \end{aligned}$$

Since A is arbitrary, we know $\sin(\sqrt{-\lambda}\pi = 0$ so $\sqrt{-\lambda}\pi = n\pi$ for positive integers n. Then

$$n = \sqrt{-\lambda}$$

Therefore we have

$$X_n(x) = A_n\sin(nx)$$

Now for the last case, we have a nontrivial solution. Let's use that to solve the T equations.

We have $-\lambda = n^2$ so from $T_{tt} - \lambda c^2 T = 0$ above

$$T_{ntt} + n^2c^2T_n = 0$$

so we have roots $\pm nci$ if we solve the characteristic equation. Then

$$T_n = C_n\sin(nct) + D_n\cos(nct)$$

So for each positive integer n,

$$X_n(x)T_n(x) = A_n\sin(nx)[C_n\sin(nct) + D_n\cos(nct)]$$

Then we have u from the infinite summation

$$u(x,t) = \sum_{n=1}^{\infty} [E_n \sin(nx)\sin(nct) + F_n \sin(nx)\cos(nct)]$$

We have two initial conditions: the profile starts at rest and the initial position.

$$\dot{u}(x,0) = 0 \text{ and } u(x,0) = \begin{cases} x & 0 \le x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x \le \pi \end{cases}$$

Using the first initial condition,

$$\begin{aligned} \dot{u}(x,0) &= \sum_{n=1}^{\infty} [E_n \sin(nx)\cos(nct)nc - F_n \sin(nx)\sin(nct)nc]\Big|_{t=0} \\ 0 &= \sum_{n=1}^{\infty} E_n \sin(nx)nc \\ E_n &= 0 \end{aligned}$$

Then

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(nx)\cos(nct)$$

From the second initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} F_n \sin(nx)$$

We can solve the Fourier series to obtain $u(x,t)$.

7.3 Laplace Equation

$u(x,y)$ is the steady state temperature in a 2-D environment. The Laplace Equation gives us

$$u_{xx} + u_{yy} = 0$$

We'll solve for the temperature in an infinite bar in the x direction and existing from $0 < y < \pi$.

Our boundary conditions are $u(x,\pi) = 0$, $u(x,0) = 0$ and $u(x,y) = 0$ as $x \to \infty$.

Our initial conditions are $u(0,y) = y^2 - \pi y$.

We'll assume

$$u(x,y) = F(x)G(y)$$

so $u_{xx} = F_{xx}G$ and $u_{yy} = FG_{yy}$. Then substituting into our Laplace equation,

$$\begin{aligned} F_{xx}G + FG_{yy} &= 0 \\ \frac{F_{xx}}{F} &= -\frac{G_{yy}}{G} = \sigma \end{aligned}$$

Then for some constant σ,

$$F_{xx} - \sigma F = 0 \text{ and } G_{yy} - \sigma G = 0$$

With our boundary conditions,

$$\begin{aligned} u(x,0) = 0 \Rightarrow F(x)G(0) &= 0 \\ G(0) &= 0 \\ u(x,\pi) = 0 \Rightarrow F(x)G(\pi) &= 0 \\ G(\pi) &= 0 \end{aligned}$$

We'll solve for G first.

When $\sigma = 0$,

$$G_{yy} = 0 \Rightarrow G = Ay + B = 0$$

And our boundary conditions give us $A = B = 0$ so discard this.

Now consider when $\sigma < 0$

$$G(y) = Ae^{\sqrt{\lambda}y} + Be^{-\sqrt{\lambda}y}$$

Using our boundary conditions, $A = B = 0$ so discard also.

Consider when $\sigma > 0$. We have complex roots at $\pm\sqrt{\sigma}i$.

$$G(y) = A\cos(\sqrt{\sigma}y) + B\sin(\sqrt{\sigma}y)$$

Our boundary conditions give

$$\begin{aligned} G(0) = 0 \Rightarrow 1A + 0B &= 0 \\ A &= 0 \\ G(\pi) = 0 \Rightarrow B\sin(\sqrt{\sigma}\pi) &= 0 \\ \sqrt{\sigma} &= n && \text{(for positive integers } n\text{)} \end{aligned}$$

Then for positive integers n,

$$G_n(y) = B_n\sin(ny)$$

Now, $F_{nxx} - n^2 F_n = 0$ for $n = 1, 2, 3....$ There are two real roots to the CE at $\pm n$. So

$$F_n(x) = C_n e^{nx} + D_n e^{-nx}$$

Therefore

$$\begin{aligned} u_n(x, y) &= F_n(x) G_n(y) \\ &= (C_n e^{nx} + D_n e^{-nx})(B_n \sin ny) \end{aligned}$$

Let $u(x, y) = \sum_{n=1}^{\infty}(C_n e^{nx} + D_n e^{-nx})(B_n \sin ny)$. Now $\lim_{x\to\infty} u(x, y) = 0$ so $C_n = 0$. Therefore

$$u(x, y) = \sum_{n=1}^{\infty} N_n e^{-nx} \sin ny$$

where $N_n = D_n B_n$.

Look at $x = 0$

$$\begin{aligned} u(0, y) &= y(y - \pi) \\ &= \sum_{n=1}^{\infty} N_n \sin ny \ e^0 \\ &= \sum_{n=1}^{\infty} N_n \sin ny \end{aligned}$$

for $0 \leq y \leq \pi$. This is a Fourier sine series.

$$\begin{aligned} N_n &= \frac{2}{\pi} \int_0^{\pi} y(y - \pi) \sin ny \ dy \\ &= \frac{4}{\pi n^3}((-1)^n - 1) \\ &= \begin{cases} \frac{-8}{\pi n^3} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} \end{aligned}$$

Then we have

$$u(x, y) = \sum_{i=1}^{\infty} e^{-(2i-1)x} \sin((2i - 1)y) \frac{-8}{\pi(2i - 1)^3}$$

Example 7.2 Heat equation on a rod going from 0 to π insulated on the ends so that $\frac{\partial u}{\partial x} = 0$ for $x = 0, \pi$. $u(x, t)$ is the temperature. The heat equation gives

$$\alpha^2 u_x x = u_t$$

For simplicity, let $\alpha = 1$ (α is normally related to the heat conductance). Assumute an initial temperature distribution of $u(x, 0) = \sin x$ for $0 \leq x \leq \pi$.

Assume separable so $u = F(x)G(t)$. Then $u_x x = F_{xx}G$ and $u_t = FG_t$.

$$
\begin{aligned}
F_{xx}G &= FG_t \\
\frac{F_x x}{F} &= \frac{G_t}{G} = \lambda
\end{aligned}
$$

where λ is a constant. Therefore $F_{xx} - \lambda F = 0$ and $G_t - \lambda G = 0$.

Remember $u_x(0,t) = u_x(\pi,t) = 0$ so $F_x(0) = F_x(\pi) = 0$. Then we solve for $F(x)$ first.

If $\lambda = 0$ then $F_{xx} = 0$ so $F(x) = Ax + B$. Then $F_x(x) = A$. Since $F_x(0) = F_x(\pi) = 0$, $A = 0$. B is arbitrary so $F(x) = B$. As well $G_t = 0$ so $G(t)$ is a constant. Then for $\lambda = 0$, we get a constant ($u(x,t) = C$).

If $\lambda > 0$ we have

$$
\begin{aligned}
F(x) &= Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} \\
F_x(x) &= A\sqrt{\lambda}e^{\sqrt{\lambda}x} - B\sqrt{\lambda}e^{-\sqrt{\lambda}x}
\end{aligned}
$$

Substituting into boundary condition, we will find that $A = B = 0$. This is the trivial solution so discard.

If $\lambda < 0$ we get complex roots

$$
\begin{aligned}
F(x) &= A\cos\sqrt{-\lambda}x \;\; + B\sin\sqrt{-\lambda}x \\
F_x(x) &= -A\sqrt{-\lambda}\sin\sqrt{-\lambda}x \;\; + B\sqrt{-\lambda}\cos\sqrt{-\lambda}x
\end{aligned}
$$

Now substituting our boundary conditions,

$$
\begin{aligned}
F_x(0) &= B\sqrt{-\lambda} \\
F_x(0) &= 0 \\
B &= 0 \\
F_x(\pi) &= -A\sqrt{-\lambda}\sin\sqrt{\underbrace{-\lambda}_{n}\,\pi} \\
F_x(\pi) &= 0 n^2 \qquad\qquad\qquad\qquad = -\lambda
\end{aligned}
$$

Look at $G_t - \lambda G = 0$ or $G_{nt} + n^2 G_n^{=} 0$.

$$
\begin{aligned}
G_n(t) &= C_n e^{-n^2 t} \\
u(x,t) &= d_0 + \sum_{n=1}^{\infty} d_n \cos nx \; e^{-n^2 t}
\end{aligned}
$$

This is a cosine series. The last thing is to consider the initial conditions.

$$\begin{aligned} u(x,t)\Big|_{t=0} &= d_0 + \sum_{n=1}^{\infty} d_n \cos nx\ e^{-n^2 t}\Big|_{t=0} \\ &= \sin x d_0 + \sum_{n=1}^{\infty} d_n \cos nx && = \sin x \end{aligned}$$

Note that d_0 is the average value

$$d_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\ dx = \frac{2}{\pi}$$

Also

$$\begin{aligned} d_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\ dx \\ &= \frac{2}{\pi}\frac{\overbrace{\cos \pi n}^{(-1)^n}+1}{1-n^2} \end{aligned}$$

Then we have

$$u(x,t) = \frac{2}{\pi} + \sum_{i=1}^{\infty} \frac{2}{\pi}\frac{2}{1-(2i)^2}\cos(2ix)e^{-(2i)^2 t}$$