MATH 239 Notes

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Gabriel Wong
me@gabrielwong.net

From lectures by Peter Nelson

Contents

1 Some Concepts

1.1 Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.2 Product of Polynomial

$$\begin{aligned}
A(x)B(x) &= \left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) \\
&= \sum_{i\geq 0}\sum_{j\geq 0} a_i b_j x^{i+j}, \quad \text{now let } k = i \text{ and } n = i+j \\
&= \sum_{n\geq 0}\left(\sum_{k\geq 0}^{n} a_k b_{n-k}\right) x^n
\end{aligned}$$

Or equivalently

$$[x^n]A(x)B(x) = \sum_{k\geq 0}^{n} a_k b_{n-k}$$

1.3 Sum Lemma

If S is a set with weight function w and A, B are sets so that $A \cap B = \varnothing$, $A \cup B = S$, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.

1.4 Product Lemma

If A, B be sets with weight function α, β respectively. Then $\Phi_A(x)\Phi_B(x) = \Phi_S(x)$ where $S = A \times B$ and $w(a,b) = \alpha(a) + \beta(b)$ is the weight function on S.

1.5 Negative Binomial Theorem

$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

equivalently

$$[x^n](1-x)^{-k} = \binom{n+k-1}{k-1}$$

2 Counting Combinations

2.1 Intro using Fruit

In how many ways can you eat n pieces of fruit given that you must eat

- at most 5 apples
- at least 3 bananas
- an even number of cherries

The answer is $[x^n]\underbrace{(1+x+x^2+x^3+x^4+x^5)}_{\text{apples}}\underbrace{(x^3+x^4+x^5+...)}_{\text{bananas}}\underbrace{(1+x^2+x^4+...)}_{\text{cherries}}$.

$$\begin{aligned}
&= [x^n]\left(\frac{1-x^6}{1-x}\right)\left(\frac{x^3}{1-x}\right)\left(\frac{1}{1-x^2}\right) \\
&= [x^n]\frac{x^3(1-x^6)}{(1-x)^3(1+x)}
\end{aligned}$$

Counting problems involving multiple selections can be encoded as coefficients. We'll now make this formal.

2.2 Sum Lemma

If S is a set with weight function w and A, B are sets so that $A \cap B = \varnothing$, $A \cup B = S$, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.

2.3 Product Lemma

If A, B be sets with weight function α, β respectively. Then $\Phi_A(x)\Phi_B(x) = \Phi_S(x)$ where $S = A \times B$ and $w(a, b) = \alpha(a) + \beta(b)$ is the weight function on S.

2.3.1 Example Proving Binomial Theorem

Let $S = \{\text{subsets of } [n]\}$ and $w(A) = |A|$ for $A \in S$. So

$$\begin{aligned}
\Phi_S(x) &= \sum_{k\geq 0}(\text{\# elements of S of weight k})\, x^k \\
&= \sum_{k\geq 0}\binom{n}{k}x^k
\end{aligned}$$

We will show inductively that this is $(1+x)^n$.

Base case $(1+x)^0 = \binom{0}{0}x^0$ is trivial. Suppose it is true for $n-1$ with $n \geq 1$.

Let $T = \{\text{elements of S containing n}\} = \{Y \cup \{n\} : Y \subseteq [n-1]\}$ and $R = \{\text{elements of } S \text{ not containing } n\} = \{Y : Y \subseteq [n-1]\}$. Clearly $T \cap R = \varnothing$. So by the Sum Lemma, $\Phi_S(x) = \Phi_R(x) + \Phi_T(x)$.

$$\begin{aligned}
\Phi_R(x) &= \sum_{Y \subseteq [n-1]} x^{|Y|} \\
&= \sum_{k \geq 0} \binom{n-1}{k} x^k \\
&= (1+x)^{n-1} \\
\Phi_T(x) &= \sum_{Y \subseteq [n-1]} x^{|Y \cup \{n\}|} \\
&= \sum_{Y \subseteq [n-1]} x^{|Y|+1} \\
&= x \sum_{Y \subseteq [n-1]} x^{|Y|} \\
&= x(1+x)^{n-1}
\end{aligned}$$

So $\Phi_S(x) = (1+x)^{n-1} + x(1+x)^{n-1} = (1-x)^n$.

2.4 Example with Fruit

For ≤ 5 apples, ≥ 3 blueberries and even number of cherries,

$$\begin{aligned}
A &= \{0, 1, 2, 3, 4, 5\} \\
B &= \{3, 4, 5, 6, ...\} \\
C &= \{0, 2, 4, 6, ...\} \\
\Phi_A(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 \\
&= \frac{1-x^6}{1-x} \\
\Phi_B(x) &= x^3 + x^4 + x^5 + x^6 + ... \\
&= \frac{x^3}{1-x} \\
\Phi_C(x) &= 1 + x^2 + x^4 + x^6 + ... \\
&= \frac{1}{1-x^2}
\end{aligned}$$

So the Product Lemma gives $\Phi_S(x) = \Phi_A(x)\Phi_B(x)\Phi_C(x)$ where $S = A \times B \times C$ and $w(a,b,c) = w(a) + w(b) + w(c)$. Then the number of valid selections for n pieces of fruit is $[x^n]\Phi_S(x)$.

$$\begin{aligned}
[x^n]\Phi_S(x) &= [x^n]\Phi_A(x)\Phi_B(x)\Phi_C(x) \\
&= [x^n]\frac{1-x^6}{1-x}\frac{x^3}{1-x}\frac{1}{1-x^2} \\
&= [x^n]\frac{x^3(1-x^6)}{(1-x)^3(1+x)} \\
&= [x^{n-3}]\frac{1-x^6}{(1-x)^3(1+x)}
\end{aligned}$$

2.5 Example of Change for \$1

Q: How many ways to make change for \$1?

A change of \$1 is a selection $(a,b,c,d) \in (\mathbb{N}_0)^4$ such that $5a + 10b + 25c + 100d = 100$. Let

$$\begin{aligned}
w_1(a) &= 5a \\
w_2(b) &= 10b \\
w_3(c) &= 25c \\
w_4(d) &= 100d
\end{aligned}$$

$$\Phi^{w}_{\mathbb{N}_0^4}(x) = \Phi^{w_1}_{\mathbb{N}_0}(x)\Phi^{w_2}_{\mathbb{N}_0}(x)\Phi^{w_3}_{\mathbb{N}_0}(x)\Phi^{w_4}_{\mathbb{N}_0}(x)$$

2.6 Negative Binomial Theorem

Prop:

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$$

equivalently

$$[x^n](1-x)^{-k} = \binom{n+k-1}{k-1}$$

Proof:

$$\begin{aligned}
[x^n](1-x)^k &= [x^n]\left(\frac{1}{1-x}\right)^k \\
&= [x^n]\underbrace{(1+x+x^2+...)(1+x+x^2+...)...(1+x+x^2+...)}_{\text{k times}}
\end{aligned}$$

This coefficient is the number of solutions to $a_1 + a_2 + ... + a_k = n$ where $a_i \in \mathbb{N}$. We show this with the product lemma. We have $\Phi_{\mathbb{N}_0}(x) = 1 + x + x^2 + x^3 + ...$ with respect to the weight function $w(a) = a$.

$$\begin{aligned}(1 + x + x^2 + ...)^k &= (\Phi_{\mathbb{N}_0}(x))^k \\ &= \Phi_S(x)\end{aligned}$$

Where $S = (\mathbb{N}_0)^k$ and $w = (a_1, a_2, ...a_k) = a_1 + a_2 + ... + a_k$.

Let $T = \{(a_1, a_2, ..., a_k) \in \mathbb{N}_0^k | a_1 + a_2 + ... + a_k = n\}$ and
$R = \{\text{Binary strings of length } n + k - 1 \text{ with exactly } k - 1 \text{ ones}\}$.

We know $|T| = [x^n](1 - x)^{-k}$ and $|R| = \binom{n+k-1}{k-1}$.

We define a bijection $f : T \to R$ by

$$f(a_1, a_2, ..., a_k) = \underbrace{0...0}_{a_1}1\underbrace{0...0}_{a_2}1...1\underbrace{0...0}_{a_k}$$

and it's inverse by

$$f(\underbrace{0...0}_{a_1}1\underbrace{0...0}_{a_2}1...1\underbrace{0...0}_{a_k}) = (b_1, b_2, ..., b_k)$$

Clearly f and g are inverses so f is a bijection and $|T| = |R|$. $\square$

We can use the negative binomial theorem to go between rational expressions and power series.
eg.

$$\begin{aligned}(1 + 2x^2)^{-5} &= \sum_{n\geq 0} \binom{n+4}{4}(-2x^2)^n \\ &= \sum_{n\geq 0}(-2)^n\binom{n+4}{4}x^{2n}\end{aligned}$$

2.7 Compositions

The ideas in the negative binomial theorem proof allude to a new type of combinatorial object.

Let $n \in \mathbb{N}_0, k \in \mathbb{N}_0$. A **composition** of n into k parts is a k-tuple $(a_1, a_2, ...a_k)$ such that $a_1 + a_2 + ... + a_k = n$ and $a_i \in \mathbb{N}$.

Example. The compositions of 5 into 3 parts are $(1, 1, 3), (1, 3, 1), (3, 1, 1,), (1, 2, 2,), (2, 1, 2,), (2, 2, 1)$. Note that order matters. Ignoring order, we have **partitions** which are much harder to work with.

Prop.
There are $\binom{n-1}{k-1}$ compositions of n with k parts.

Proof.
Let $S = \{\text{Compositions of } n \text{ into } k \text{ parts}\}$, $T = \{\text{solutions to } a_1 + a_2 + ... + a_n \text{ with } a_i \in \mathbb{N}_0\}$.

$f(a_1, a_2, ..., a_k) = (a_1 - 1, a_2 - 1, ..., a_k - 1)$ gives a bijection from S to T. By the material in the proof earlier,

$$|T| = \binom{(n - k + k - 1)}{k - 1}$$
$$|T| = |S| = \binom{n-1}{k-1}$$

□

Prop.
The number of compositions of n into any number of parts is 2^{n-1}.

Proof.
By previous proposition, the number is $\sum_{k\geq 1} \binom{n-1}{k-1} = 2^{n-1}$ by the binomial theorem.

2.8 Restricted Compositions

Often we will need to compute the number of compositions of n with various restrictions on the number of parts, or their sizes. The sum/product lemmas do this.

2.8.1 Small Parts

How many compositions of n have each part equal to 1 or 2.

- With k parts?
- With any number of parts?

Let $S = \{1, 2\}$ and $w(\sigma) = \sigma$ for each $\sigma \in S$. Then $\Phi_S(x) = x + x^2$.

Consider $[x^n]\Phi_S(x)^k$. By the product lemma, it is equal to the number of k-tuples $(a_1, a_2, ..., a_k) \in S^k$ with $a_1 + a_2 + ... + a_k = n$. So this is the number of compositions of n into k parts of size 1 or 2.

$$\begin{aligned}
[x^n]\Phi_S(x)^k &= [x^n](x + x^2)^k \\
&= [x^n]x^k(1 + x)^k \\
&= [x^{n-k}](1 + x)^k \\
&= \binom{k}{n-k}
\end{aligned}$$

So the number of compositions of n into k parts of size 1 or 2 is $\binom{k}{n-k}$. So the number of compositions of n into any number of parts of size 1 or 2 is $\sum_{k\geq 0}\binom{k}{n-k}$.

Alternatively, the number of compositions of n into any number of parts of size 1 or 2 is

$$\begin{aligned}\sum_{k\geq 0}[x^n](x+x^2)^k &= [x^n]\sum_{k\geq 0}(x+x^2)^k \\ &= [x^n]\frac{1}{1-x-x^2} \\ &= n\text{th Fibonacci number}\end{aligned}$$

2.8.2 Odd Parts

How many compositions of n have each part odd?

Let $S=\{1,3,5,7,...\}$ and $w(\sigma)=\sigma$ for each $\sigma\in S$.

$$\begin{aligned}\Phi_S(x) &= x^1+x^3+x^5+x^7+... \\ &= x(1+x^2+x^4+...) \\ &= \frac{x}{1-x^2}\end{aligned}$$

Then the number of compositions of n into k odd parts is $[x^n]\Phi_S(x)^k$. So the number of compositions of n into any number of odd parts is

$$\begin{aligned}\sum_{k\geq 0}[x^n]\Phi_S(x)^k &= [x^n]\sum_{k\geq 0}\Phi_S(x)^k \\ &= [x^n]\frac{1}{1-\Phi_S(x)} \\ &= [x^n]\frac{1}{1-\frac{x}{1-x^2}} \\ &= [x^n]\frac{1-x^2}{1-x-x^2}\end{aligned}$$

Let $\frac{1-x^2}{1-x-x^2}=a_0+a_1x+a_2x^2+...$

Solving $(1-x-x^2)(a_0+a_1x+a_2x^2+...)=1-x^2$ we get

$$\begin{aligned}a_0 &= 1 \\ a_1-a_0 &= 0 \\ a_2-a_1-a_0 &= -1 \\ a_k-a_{k-1}-a_{k-2} &= 0, \quad k\geq 3\end{aligned}$$

Then $a_0 = 1, a_1 = 1, a_2 = 1$ and $a_k = a_{k-1} + a_{k-2}$ for $k \geq 3$. So the number of compositions of n into odd parts is the $(n-1)$th Fibonacci number.

2.8.3 Combinatorial Proof of Compositions of Size 1 and 2

Let $A_n = \{$compositions of n into parts of size 1 or 2$\}$. We need $|A_n| = |A_{n-1}| + |A_{n-2}|$. Let $A'_n = \{$compositions of n into parts of size 1 or 2 with last part 1$\}$. Let $A''_n = \{$compositions of n into parts of size 1 or 2 with last part 2$\}$.

Let $f_1 : A'_n \to A_{n-1}$ be defined by $f(a_1, a_2, ..., a_k) = (a_1, a_2, ...a_{k-1}$. Its inverse is $f_1^{-1} : A_{n-1} \to A'_n$ defined by $f^{-1}(b_1, b_2, ..., b_k) = (b_1, ..., b_k, 1)$. A similar bijection can be found between A''_n and A_{n-2}.

So since $|A_n| = |A'_n| + |A''_n|$, $|A_n| = |A_{n-1}| + |A_{n-2}|$. $\square$.

2.8.4 Combinatorial Proof of Odd Sized Compositions

Let $T_n = \{$compositions of n into parts of odd size $\}$. Clearly $|T_1| = |T_2| = 1$. To show that T_{n+1} is the nth Fibonacci number, it suffices to show that $|T_n| = |T_{n-1}| + |T_{n-2}|$ for $n \geq 3$.

We do this by defining a bijection f between T_n and $T_{n-1} \cup T_{n-2}$.

$$\begin{aligned}
&T_2 = \{(1,1)\} \\
&T_3 = \{(1,1,1),(3)\} \\
&T_4 = \{(1,1,1,1),(1,3),(3,1)\} \\
&T_5 = \{(1,1,1,1,1),(1,1,3),(1,3,1),(3,1,1),(5)\}
\end{aligned}$$

Let $f : T_n \to T_{n-1} \cup T_{n-2}$ be defined by

$$f(a_1, a_2, ..., a_k) = \begin{cases} (a_1, a_2, ..., a_{k-1}) & a_k = 1 \\ (a_1, a_2, ..., a_k - 2) & a_k \neq 1 \end{cases}$$

and $g : T_{n-1} \cup T_{n-2} \to T_n$ be defined by

$$g(a_1, a_2, ..., a_k) = \begin{cases} (a_1, a_2, ..., a_k, 1) & (a_1, ..., a_k) \in T_{n-1} \\ (a_1, a_2, ..., a_k + 2) & (a_1, ..., a_k) \in Tn - 2 \end{cases}$$

Then g is the inverse of f. So f is a bijection and thus $|T_n| = |T_{n-1} \cup T_{n-2} = |T_{n-1}| + |T_{n-2}|$.

$\square$

2.8.5 Relationship between Above Compositions

Let $T_n = \{$compositions of n into parts of odd size$\}$. Let $S_n = \{$compositions of n into parts of size 1 or 2$\}$. We'll show that $|S_n| = |T_{n+1}|$ by finding a bijection.

We have $(1, 3, 7, 5, 9, 3, 3, 1, 3) \in T_{35}$ can be mapped to

$$(\underbrace{1}_{1}, \underbrace{2,1}_{3}, \underbrace{2,2,2,1}_{7}, \underbrace{2,2,1}_{5}, \underbrace{2,2,2,2,1}_{9}, \underbrace{2,1}_{3}, \underbrace{2,1}_{3}, \underbrace{1}_{1}, \underbrace{2,1}_{3})$$

This is done by transforming each element in the composition as a 1 prefixed by the appropriate number of 2s.

However, this results in compositions that always end in 1. So we remove the final 1 to map T_{35} to S_{34}. This rule can be formally defined as a bijection so $|T_{n+1}| = |S_n|$. $\square$

3 Binary Strings

A binary string of length k is a k-tuple $(a_1, ..., a_k)$ where $a_i \in \{0, 1\}$. Equivalently, a member of $\{0, 1\}^k$. We usually supress commas and brackets and write strings as $a_1a_2...a_n$.

If $\sigma = s_1s_2...s_j$ and $\tau = t_1t_2...t_k$ then $\sigma\tau = s_1s_2...s_jt_1t_2...t_k$. (**concatenation**)

We write $l(\sigma)$ for the length of σ. So $l(\sigma\tau) = l(\sigma) + l(\tau)$.

σ^k denotes $\underbrace{\sigma\sigma...\sigma}_{k \text{ times}}$ and $\sigma^0 = \epsilon$.

If A, B are sets of strings then $AB = \{\alpha\beta : \alpha \in A, \beta \in B\}$.

We also define $A^k = \underbrace{AAA...A}_{k \text{ times}}$.

Example 3.1 $\{0, 1\}^7 = \{\text{strings of length } 7\}$

$$\begin{aligned} A^* &= \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup ... \\ &= \bigcup_{k \geq 0} A^k \end{aligned}$$

A **substring** of s is a string b such that $s = abc$ for some a, c.

A **block** of s is a maximal substring of s whose members are equal (ie. all 0 or 1).

3.1 Ambiguity

If each such string in A^* can only be optained from A^* in one way, then A^* is **unambiguous**. Other expressions can also be called ambiguous or unambiguous.

For example, $\{0, 00\}\{0, 00, 000\}$ is ambiguous since 000 can be made in multiple ways. $\{0, 1\}$ is unambiguous. Also for any set A such that $\epsilon \in A$, A^* is ambiguous.

Is $\{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^*$ ambiguous? No. It is unambiguous but generates all possible binary strings. We can decompose any string by taking all 1s in the front and 0s in the back into $\{1\}^*$ and $\{0\}^*$. $\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*$ captures blocks of 0s and 1s in the middle.

Another unambiguous expression generating all binary strings is $\{0, 1\}^*$. However, it is less useful than the previous expression for counting problems.

3.2 Strings and Generating Series

Let S be a set of binary strings with $w(\sigma) = \text{length}(\sigma)$. Then the number of strings of length n in S is $[x^n]\Phi_S(x)$.

Theorem 3.1 *If $S = A \cup B$* ***unambiguously****, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.*

If $S = AB$ ***unambiguously****, then $\Phi_S(x) = \Phi_A(x)\Phi_B(x)$.*

If $S = A^*$ ***unambiguously****, then $\Phi_S(x) = \frac{1}{1-\Phi_S(x)}$. Notice that $\Phi_S(x)$ must have a zero constant term, which agrees with the fact that A is ambiguous if it contains ϵ.*

Example 3.2 Let $S = \{\text{binary strings where each block of zero has even length}\}$.

We know $S = \{00, 1\}^*$ unambiguously. Then the number k of strings of length n in S is

$$\begin{aligned} k &= [x^n]\Phi_S(x) \\ &= [x^n]\frac{1}{1-\Phi_A(x)} \end{aligned}$$

where $A = \{00, 1\}$. $\Phi_A(x) = x + x^2$ so $[x^n]\Phi_S(x) = [x^n]\frac{1}{1-x-x^2}$. Therefore the answer is the nth Fibonacci number.

Example 3.3 Let $S = \{\text{strings with exactly three blocks}\}$.

We can decompose S as $S = \underbrace{\{\{1\}\{1\}^*\{0\}\{0\}^*\{1\}\{1\}^*\}}_{A_1} \cup \underbrace{\{\{0\}\{0\}^*\{1\}\{1\}^*\{0\}\{0\}^*\}}_{A_0}$. That is, $S = \{\text{strings of the form } 1...10...01...1\} \cup \{\text{strings of the form } 0...01...10...0\}$.

$$\begin{aligned} \Phi_{A_1}(x) &= \Phi_{\{1\}}(x)\Phi_{\{1\}^*}(x)\Phi_{\{0\}}(x)\Phi_{\{0\}^*}(x)\Phi_{\{1\}}(x)\Phi_{\{1\}^*}(x) \\ &= (x)\left(\frac{1}{1-x}\right)(x)\left(\frac{1}{1-x}\right)(x)\left(\frac{1}{1-x}\right) \\ &= \frac{x^3}{(1-x)^3} \end{aligned}$$

Similarly $\Phi_{A_0}(x) = \frac{x^3}{(1-x)^3}$.

$$\begin{aligned}\Phi_S(x) &= \Phi_{A_0}(x) + \Phi_{A_1}(x) \\ &= \frac{2x^3}{(1-x)^3} \\ &= 2x^3 \sum_{n\geq 0} \binom{n+2}{2} x^n\end{aligned}$$

So the number of elements in S of length n is $2\binom{n-1}{2}$.

This makes sense intuitively since we are picking two positions where the string swaps between repeating 0 and repeating 1. And the string can either start with 0 or 1.

Example 3.4 Let S be the set of strings with all blocks with length ≥ 2.

Then $S = (\epsilon \cup \{00\}0^*)(\{11\}1^*\{00\}0^*)^*(\epsilon \cup \{11\}1^*)$.

$$\begin{aligned}\Phi_S(x) &= \left(1 + \frac{x^2}{1-x}\right) \frac{1}{1 - \left(\frac{x^2}{1-x}\frac{x^2}{1-x}\right)} \left(1 + \frac{x^2}{1-x}\right) \\ &= \left(\frac{1-x+x^2}{1-x}\right)^2 \frac{(1-x)^2}{(1-x)^2 - x^4} \\ &= \frac{(1-x+x^2)^2}{(1-x)^2 - x^4} \\ &= \frac{1-x+x^2}{1-x-x^2}\end{aligned}$$

Example 3.5 Let S be the set of strings where an even block of 0s cannot be followed by an odd number of block of 1s.

$$S = 1^*\Big(\underbrace{\{0(00)^*11^*\}}_{\text{odd 0s}} \cup \underbrace{\{00(00)^*11(11)^*\}}_{\text{even 0s}}\Big)^* 0^*$$

$$\begin{aligned}\Phi_S(x) &= \frac{1}{1-x} \frac{1}{1 - \left(\frac{x}{1-x^2}\frac{x}{1-x} + \frac{x^2}{1-x^2}\frac{x^2}{1-x^2}\right)} \frac{1}{1-x} \\ &= \frac{(1+x)^2}{x(1+x^2+x^3)}\end{aligned}$$

Example 3.6 Let S be the set of strings with no l consecutive 1s and no m consecutive 0s.

$$
\begin{aligned}
S &= (0^* \setminus \{0^m0^*\}) \left[\left(\{11^*\} \setminus \{1^l1^*\}\right) \left(\{00^*\} \setminus \{0^m0^*\}\right) \right]^* \left(1^* \setminus \{1^l1^*\}\right) \\
\Phi_S(x) &= \left(\frac{1}{1-x} - \frac{x^m}{1-x} \right) \left(\frac{1}{1 - \left(\frac{x}{1-x} - \frac{x^l}{1-x}\right)\left(\frac{x}{1-x} - \frac{x^m}{1-x}\right)} \right) \left(\frac{1}{1-x} - \frac{x^l}{1-x} \right) \\
&= \frac{1 - x^m - x^l + x^{m+l}}{1 - 2x + x^{m+1} + x^{l+1} - x^{m+l}} \qquad \text{(after some algebra)}
\end{aligned}
$$

Considering $l = 1, m = 1$,

$$
\begin{aligned}
\Phi_S(x) &= \frac{1 - 2x + x^2}{1 - 2x + x^2 + x^2 - x^2} \\
&= 1
\end{aligned}
$$

This makes sense since only ϵ satisfies the constriants.

Considering $l = 2, m = 2$,

$$
\begin{aligned}
\Phi_S(x) &= \frac{1 - 2x + x^4}{1 - 2x + 2x^3 - x^4} \\
&= \frac{(1 - x^2)^2}{(1 - x^2)(1 - 2x + x^2)} \\
&= \frac{1 - x^2}{1 - 2x + x^2} \\
&= \frac{1 + x}{1 - x}
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
1 + x &= a_0(1 - x) + a_1x(1 - x) + a_2x^2(1 - x) + ... \\
a_0 &= 1 \\
-a_0 + a_1 &= 1 \Rightarrow a_1 = 2 \\
a_i - a_{i-1} &= 0 \Rightarrow a_{i+1} = a_i \forall i \geq 2
\end{aligned}
$$

This makes sense since we can have either ϵ, 0101...0101 or 1010...1010.

3.3 Recursive Decompositions

Example 3.7 Let S be the set of all strings.

S can be recursively described as $S = \{\epsilon\} \cup S\{0, 1\}$. We then have the generating function,

$$
\begin{aligned}
\Phi_S(x) &= 1 + \Phi_S(x)(2x) \\
\Phi_S(x) - 2x\Phi_S(x) &= 1 \\
\Phi_S(x) &= \frac{1}{1-2x} \\
\Phi_S(x) &= \sum_{k\geq 0} 2^k x^k
\end{aligned}
$$

Which gives us that there are 2^k binary strings of length k, as expected.

Example 3.8 Let S be the set of strings without 111.

$$
\begin{aligned}
S &= \{\epsilon, 1, 11\} \cup S\{0, 01, 011\} \\
\Phi_S(x) &= (1 + x + x^2) + \Phi_S(x)(x + x^2 + x^3) \\
\Phi_S(x) &= \frac{1 + x + x^2}{1 - (x + x^2 + x^3)}
\end{aligned}
$$

Example 3.9 How many strings are there with no 11101?

Let L be the set of strings without 11101. Let M be the set of strings with 11101 at the end and not anywhere else in the string. Notice that L and M are disjoint.

$L \cup M = \{\epsilon\} \cup L\{0, 1\}$. Adding a 0 or 1 won't add 11101 in the middle of the string but can add it to the end.

We need to find an expression for M. We don't have $M = L\{11101\}$ since $\{1110\}\{11101\}$ has two 11101 sequences.

$L\{11101\} = M \cup M\{1101\}$. This accounts for the fact that we can create a second 11101 sequence by appending to M.

$$
\begin{aligned}
\Phi_L(x) + \Phi_M(x) &= 1 + 2x\Phi_L(x) && \text{(from } L \cup M = \{\epsilon\} \cup L\{0,1\}\text{)} \\
\Phi_L(x)x^5 &= \Phi_M(x) + \Phi_M(x)x^4 && \text{(from } L\{11101\} = M \cup M\{1101\}\text{)} \\
\Phi_M(x) &= \frac{x^5}{1+x^4}\Phi_L(x) \\
\Phi_L(x) + \frac{x^5}{1+x^4}\Phi_L(x) &= 1 + 2x\Phi_L(x) && \text{(substituting into first equation)} \\
\Phi_L(x) &= \frac{1}{1 - 2x + \frac{x^5}{1+x^4}} \\
\Phi_L(x) &= \frac{1 - x^4}{1 - 2x - x^4 + 3x^5}
\end{aligned}
$$

4 Evaluating Coefficients of Generating Series

4.1 Partial Fractions

Example 4.1 Let $f(x) = \frac{1+3x}{(1-x)(1+x)(1-2x)}$

$$\begin{aligned} f(x) &= \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-2x} \\ &= \frac{A(1+x)(1-2x) + B(1-x)(1-2x) + C(1-x)(1+x)}{(1-x)(1+x)(1-2x)} \end{aligned}$$

$$\begin{aligned} A + B + C &= 1 \\ -A - C &= 3 \\ -2A + 2B + C &= 0 \end{aligned}$$

So we have $A = -2, B = -\frac{1}{3}, C = \frac{1}{3}$. Substituting,

$$\begin{aligned} f(x) &= \frac{-2}{1-x} - \frac{1}{3}\frac{1}{1+x} + \frac{10}{3}\frac{1}{1-2x} \\ &= -2\sum_{k\geq 0} x^k - \frac{1}{3}\sum_{k\geq 0}(-x)^k + \frac{10}{3}\sum_{k\geq 0}(2x)^k \\ &= \sum_{k\geq 0}(-2 - \frac{1}{3}(-1)^k + \frac{10}{3}2^k)x^k \end{aligned}$$

Then $[x^n]f(x) = -2 - \frac{1}{3}(-1)^n + \frac{10}{3}2^n$.

Theorem 4.1 *Let f, g be polynomials with $\deg(g) < \deg(f)$ and f has constant term 1. Then*

$$\frac{g(x)}{f(x)} = \frac{h_1(x)}{(1-\Theta_1(x))^{m_1}} + \frac{h_2(x)}{(1-\Theta_2(x))^{m_2}} + ... + \frac{h_l(x)}{(1-\Theta_l(x))^{m_l}}$$

with $\deg(h_i) < m_i$ $\forall i \in [l]$.

4.2 Solving Recurrences

Theorem 4.2 *Let $p(x)$ and $q(x)$ be polynomials with $\deg(p(x)) < \deg(q(x))$ and $q(x) = (1-\theta_1 x)^{m_1} ... (1-\theta_k x)^{m_k}$ where $m_1, m_2, ..., m_k \in \mathbb{N}$ and $\theta_1, \theta_2, ..., \theta_k \in \mathbb{C}$ are distinct.*

Then there exists polynomials $A_1(x), ..., A_k(x)$ with $\deg(A_1) < m_1, ..., \deg(A_k) < m_k$ such that $[x^n]\frac{p(x)}{q(x)} = A_1(n)\theta_1^n + ... + A_k(n)\theta_k^n$ for all $n \geq 0$.

Given a recurrence $a_n = q_1 a_{n-1} + q_2 a_{n-2} ... + q_k a_{n-k}$, $n \geq k$ and initial values for $a_0, a_1, ..., a_{k-1}$, determine a_n explicitly.

The **characteristic polynomial** for such a recurrence is $1 - q_1x - q_2x^2 - ... - q_kx^k$. Equivalently, it is $1 + q_1x + q_2x^2 + .. + q_kx^k$ for $a_n + q_1a_{n-1} + q_2a_{n-2}... + q_ka_{n-k} = 0$.

Theorem 4.3 *Given such a recurrence, let* $A(x) = a_0 + a_1x + a_2x^2 + ...$

Then $A(x) = \frac{p(x)}{q(x)}$ *where* q *is the characteristic polynomial and* $\deg(p) < k$.

Proof 4.1 We need to show that $A(x)q(x)$ is a polynomial with degree $< k$.

Let $n \geq k$. Then

$$\begin{aligned}
[x^n]A(x)q(x) &= [x^n](a_0 + a_1x + a_2x^2 + ...)(1 - q_1x - q_2x^2 - ... - q_kx^k) \\
&= a_n - q_1a_{n-1} - q_2a_{n-2} - ... - q_ka_{n-k} \\
&= 0 && \text{(by definition of } a_n\text{)}
\end{aligned}$$

So then $\deg(A(x)(q(x)) < k$ as required. □

Combining **Theorem 4.2 and 4.3**, we have

Theorem 4.4

$$\begin{aligned}
a_n &= [x^n]A(x) \\
&= [x^n]\frac{p(x)}{q(x)} \\
&= A_1(n)\theta_1^n + ... + A_k(n)\theta_k^n
\end{aligned}$$

where $\deg(p) < k$, *q is the characteristic polynomial,* $\theta_1, ...\theta_j$ *are distinct,* $m_1, ..., m_j \in \mathbb{N}$, $q(x) = (1 - \theta_1x)^{m_1}...(1 - \theta_jx)^{m_j}$ *and* A_i *is a polynomial of degree* $< m_i$.

Example 4.2 Solve the recurrence defined by

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -1 \\
a_2 &= 17 \\
a_n &= a_{n-1} + 8a_{n-2} - 12a_{n-3}
\end{aligned}$$

The characteristic polynomial is

$$\begin{aligned}
q(x) &= 1 - x - 8x^2 + 12x^3 \\
&= (1 - 2x)^2(1 + 3x)
\end{aligned}$$

So $\theta_1 = 2, \theta_2 = -3$ and $m_1 = 2, m_2 = 1$.

So we know that there are polynomials $A_1(x), A_2(x)$ where $\deg(A_1) < 2, \deg(A_2) < 1$ and $a_n = A_1(n)2^n + A_2(n)(-3)^n$ for all n.

Let $A_1(x) = \alpha x + \beta$ and $A_2(x) = \gamma$. Then $a_n = (\alpha n + \beta)2^n + \gamma(-3)^n$.

Using our values for a_0, a_1, a_2, we have

$$\begin{aligned} a_0 &= 1 &&= \beta + \gamma \\ a_1 &= -1 &&= 2(\alpha + \beta) - 3\gamma \\ a_2 &= 17 &&= 4(2\alpha + \beta) + 9\gamma \end{aligned}$$

$\alpha = 1, \beta = 0, \gamma = 1$ is the only solution. So $a_n = n2^n + (-3)^n$.

4.3 Binary Trees

A binary tree is either empty or a root vertex together with a left child and a right child, each of which is a (possibly empty) binary tree. This can be represented as $(\bullet, S_1, S_2)$.

Let T be the set of binary trees and $w(S) =$ the number of vertices in S for each $S \in T$. We can recursively define this as $w(\epsilon) = 0$ and $w(\bullet, S_1, S_2) = 1 + w(S_1) + w(S_2)$.

Let $T(x) = \Phi_T(x)$. Thus $[x^n]T(x)$ is the number of binary trees of n verticies.

We have $T = \{\epsilon\} \cup \{\bullet\} \times T \times T$ unambiguously. Then

$$\begin{aligned} \Phi_T(x) &= \Phi_{\{\epsilon\}}(x) + \Phi_{\{\bullet\}}(x)\Phi_T(x)^2 \\ T(x) &= 1 + xT(x)^2 \\ xT(x)^2 - T(x) + 1 &= 0 \\ 4x^2T(x)^2 - 4xT(x) + 4x &= 0 \\ (2xT(x) - 1)^2 - 1 + 4x &= 0 \\ (1 - 2xT(x))^2 &= 1 - 4x \\ 1 - 2xT(x) &= \pm\left(1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}\right) \qquad \text{(by assignment 3)} \end{aligned}$$

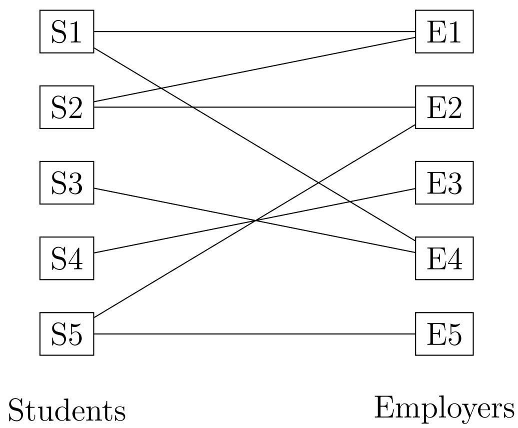
We cannot have the negative version since the LHS and the RHS would have different constant terms.

$$\begin{aligned} 1 - 2xT(x) &= 1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1} \\ T(x) &= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n \end{aligned}$$

Therefore there are $\frac{1}{n+1}\binom{2n}{n}$ binary trees on n vertices.

5 Graph Theory

- Given a circuit diagram, can we make a flat circuitboard without edges crossing? (**Planarity**)
- How many colours are needed to colour each point in the plane so that no two points at distance 1 get the same colour?
- How many ways are there to drive between two intersections in Manhattan's one way system?
- Given some SE students and coop positions, where each position is compatible with only some students, can we give everyone a job?



- What is the cheapest way to get between two given cities?

5.1 Definitions

A **graph** is a pair (V, E) where V is a finite set and E is a set of unordered pairs of distinct elements of V (ie. two-element subsets of V).

We call the elements of V the **vertices** and the elements of E the **edges**.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. An **isomorphism** from G_1 to G_2 is a bijection $\phi : V_1 \to V_2$ such that for all $u, v \in V_1$, $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$.

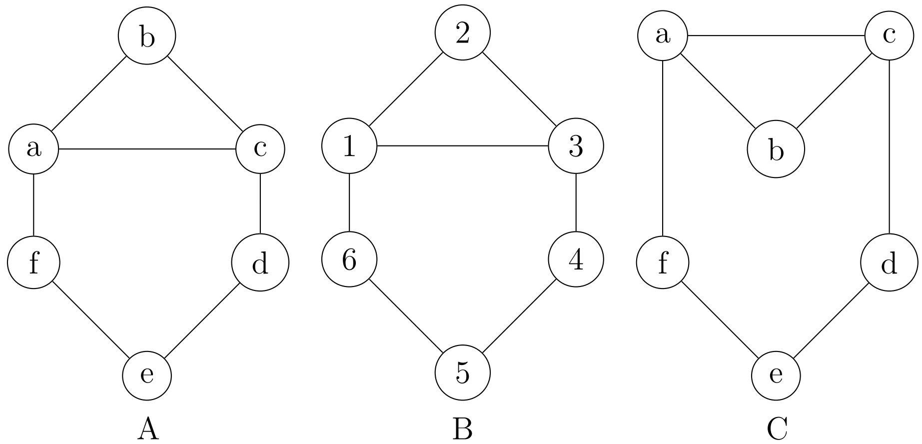
If an isomorphism exists then G_1 and G_2 are **isomorphic**. Graphs are isomorphic if they can be drawn in the same way.

We abbreviate an edge $\{u, v\}$ by uv. If $uv \in E$ then u and v are **adjacent** or **neighbours**.

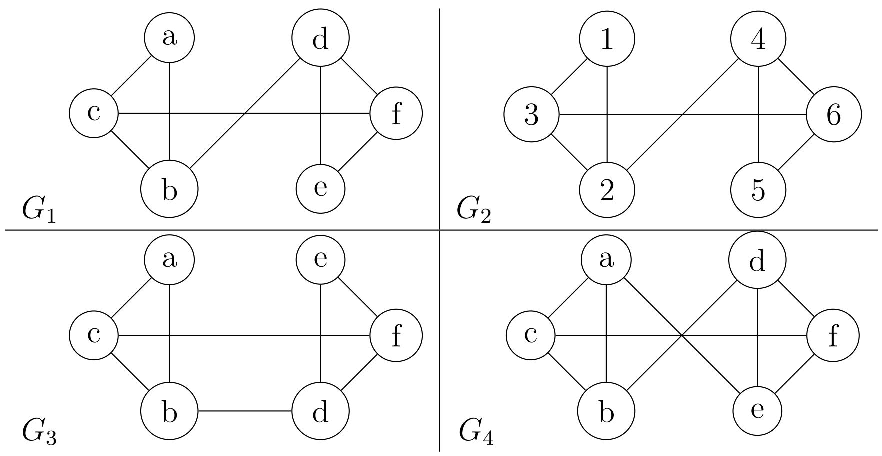
The **degree** of a vertex is its number of neighbours.

An edge uv is **incident** with vertices u and v.

Example 5.1 Graphs A and B are the equal, although drawn differently. A and B are isomorphic but are not the equal since the vertices are labelled differently.



Example 5.2 G_1 and G_3 are equal. G_2 is not equal since the vertex names are different but isomorphic to G_1 and G_3. G_4 is not equal since it has an extra edge.



Theorem 5.1 *Handshake Theorem*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Proof 5.1 Let $S = \{(v, e) : v \text{ is incident with } e\}$.

$$\begin{aligned} |S| &= \sum_{v \in V} (\text{\# edges incident with } v) \\ &= \sum_{v \in V} \deg(v) \end{aligned}$$

Also

$$\begin{aligned}|S| &= \sum_{e\in E}(\#\text{ vertices incident with } e)\\ &= 2|E|\end{aligned}$$

So $\sum_{v\in V}\deg(v) = 2|E|$. $\square$

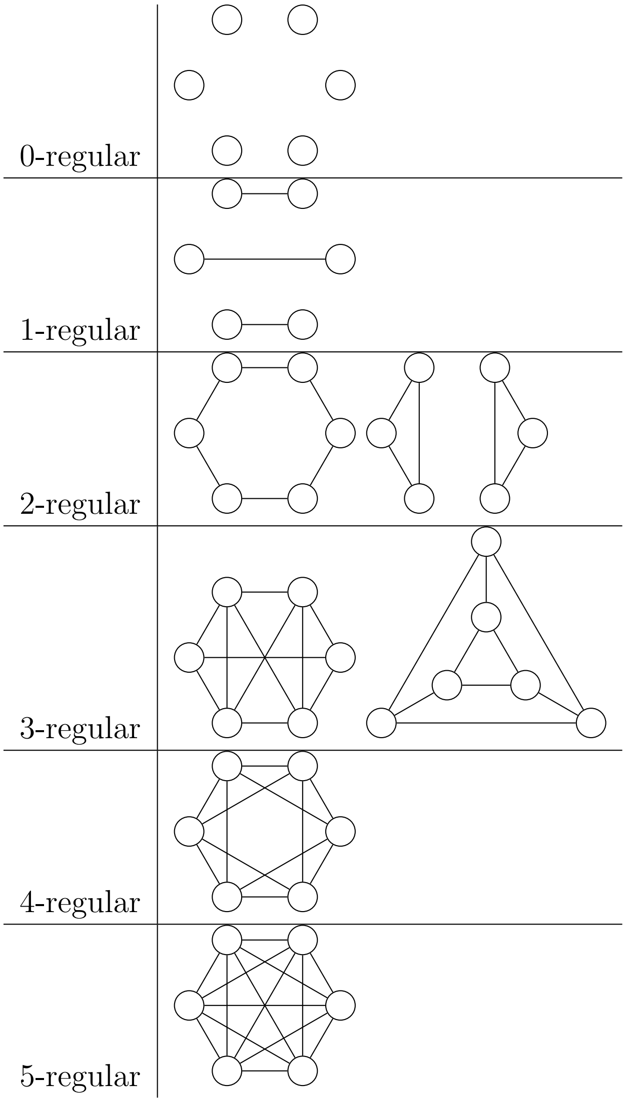
Theorem 5.2 *Every graph has an even number of verticies of odd degrees.*

This follows from the previous theorem. Since $\sum_{v\in V}\deg(v) = 2|E|$ *is even,* $\deg(v)$*is odd for an even number of* $v \in V$.

5.2 Regular Graphs

A graph is **regular** if every vertex has the same degree. If this degree is d, then the graph is called d-regular.

Example 5.3 The following table shows all d-regular 6 vertex graphs.

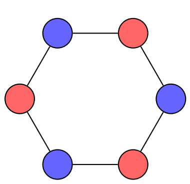


5.3 Bipartite Graph

A **bipartite graph** is a graph $G = (V, E)$ for which there exists sets A, B such that $A \cup B = V$, $A \cap B = \varnothing$ and every edge is incident with a vertex in A and a vertex in B.

(A, B) is a **bipartition** of G.

Example 5.4 A graph is bipartite if there is a 2-coloring for the graph. The following graph is bipartite since it has a 2-coloring.



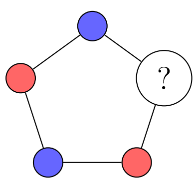
5.4 Cycle

A ***k*-cycle** is a graph $C_k = (V, E)$ so that V has an ordering $v_1, v_2, ..., v_k$ so that $E = \{v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv1\}$. So a k-cycle has k vertices and k edges.

Theorem 5.3 *A k-cycle is bipartite if and only if k is even.*

Proof 5.2 If k is even, then $(\{v_1, v_3, v_5, ..., v_{k-1}\}, \{v_2, v_4, ...v_k\})$ is a bipartition so C_k is bipartite.

If k is odd, WLOG suppose (A, B) is a bipartition with $v_1 \in A$. We show inductively that $v_i \in A$ whenever i is odd. This is true for $i = 1$. If it is true for some v_i then since $v_iv_{i+1} \in E$ and $v_{i+1}v_{i+2} \in E$, we have $v_{i+1} \in B$ and $v_{i+2} \in A$. By induction, $v_i \in A$ for all odd i. Thus $v_k \in A$ and $v_i \in A$. So since $v_kv_1 \in E$, (A, B) is not a bijection.

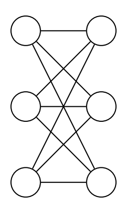


5.5 Complete Graph

A **complete graph** K_n is a graph $G = (V, E)$ so that $|V| = n$ and every pair of vertices is adjacent. A complete graph has $\binom{n}{2}$ edges.

Only K_1 and K_2 are bipartite.

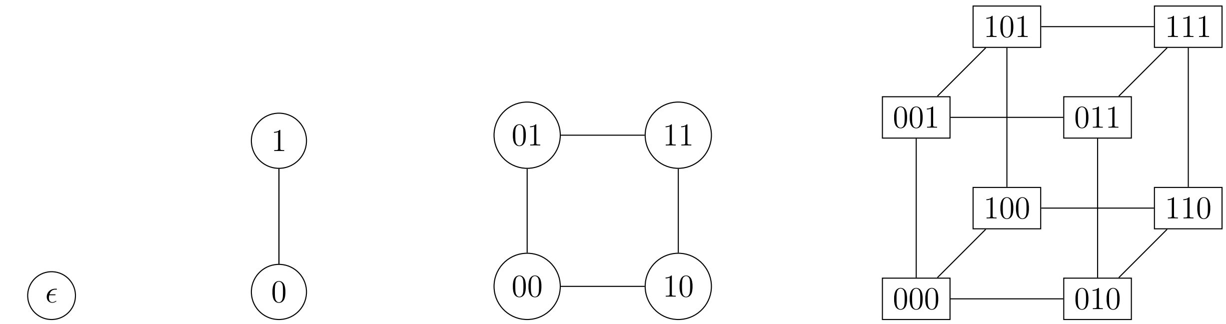
A **complete bipartite graph** $K_{m,n}$ is a bipartite graph with bipartition (A, B) so that every vertex in A is adjacent to every vertex in B and $|A| = m$ and $|B| = n$. From this, we get $K_{m,n}$ has mn edges.



$K_{3,3}$

5.6 Cube

For $n \geq 0$, an ***n*-cube** is a graph with $V = \{\text{binary strings of length } n\}$ in which two vertices are adjacent if they differ in exactly one position.



0-cube 1-cube 2-cube 3-cube

Proof 5.3 The n-cube has 2^n vertices and $n2^{n-1}$ edges.

There are 2^n vertices because there are 2^n binary strings.

For each string s of length n, there are exactly n strings that differ from s in exactly one position. So each vertex of the n-cube has degree n. By the Handshake Theorem,

$$\begin{aligned} 2|E| &= \sum_{v \in V} \deg(v) \\ 2|E| &= |V|n \\ 2|E| &= n2^n \\ |E| &= n2^{n-1} \end{aligned}$$

In general, for a d-regular graph G, we have

$$\begin{aligned} 2|E| &= \sum_{v \in V} \deg(v) \\ 2|E| &= d|V| \\ |E| &= \frac{d|V|}{2} \end{aligned}$$

The n-cube can be constructed recursively from the $(n-1)$-cube by taking two copies of the $(n-1)$-cube and joining pairs of corresponding vertices with an edge.

Proof 5.4 The n-cube is bipartite for all n.

Given a string with an even number of 1s, every neighbour will have an odd number of 1s. Therefore ({strings of length n with an even number of 1s}, {strings of length n with an odd number of 1s}) is a bipartition of the n-cube for any n.

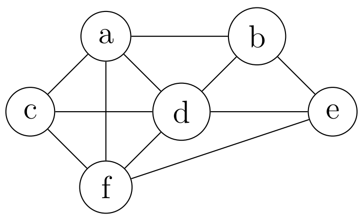
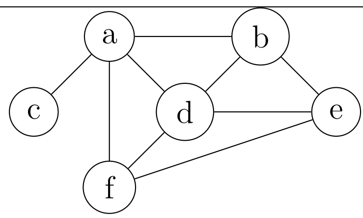
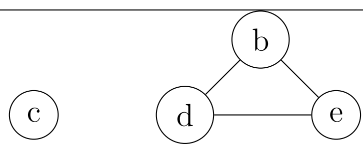
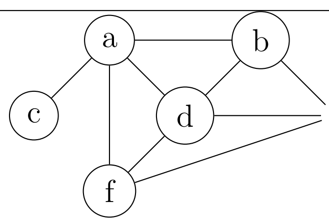
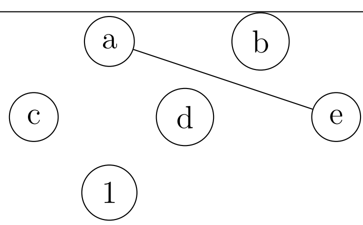
End of midterm material ←

5.7 Subgraph

A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. Essentially, it is a graph obtained by removed any number of edges or vertices from G.

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a **spanning subgraph** of G if $V' = V$.

Example 5.5

G		Original Graph
H_1		Subgraph of G
H_2		Subgraph of G
H_3		Not a subgraph of G. It is not a graph due to edges to a nonexisting vertex.
H_4		Not a subgraph of G. Contains $1 \in V'$ but $1 \notin V$ and $(a, e) \in E'$ but $(a, e) \notin E$.

5.8 Walk

A **walk** of a graph G is an alternating series of vertices and edges $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k$ so that $v_0, v_1, ..., v_k \in V$ and each e_i is an edge of G from v_{i-1} to v_i. The **length** of this walk is k, or the number of edges.

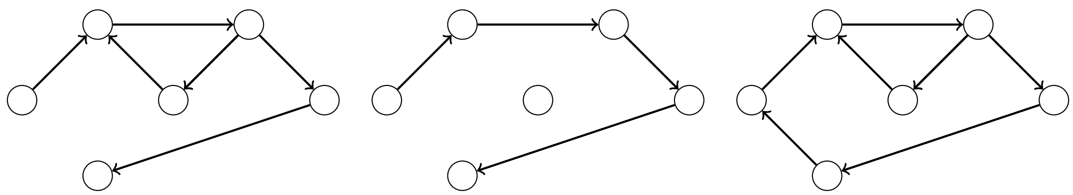
If $v_0, v_1, ..., v_k$ are distinct, the walk is also a **path**.

If $v_0, v_1, ...v_k$ is a walk and $v_0 = v_k$, then the walk is **closed**.

If $v_0, v_1, ..., v_k$ is a closed walk and $v_0, v_1, ..., v_{k-1}$ are distinct, then the walk is a **cycle**.

A cycle that contains every vertex of a graph G is a **Hamilton cycle**. A graph with a Hamilton Cycle is **Hamiltonian**.

To specify a walk (or path) we often just list its vertices.



Walk Path Closed Walk

Proof 5.5 If there is a walk from x to y in G, then there is also a path.

Let $x = v_0, v_1, ..., v_k = y$ be a **shortest** walk from x to y in G.

We argue that this walk is actually a path. Suppose it is not a path. Then there exists i, j such that $0 \leq i < j \leq k$ and $v_i = v_j$.

But then $v_0, v_1, ..., v_i, v_{j+1}, v_{j+2}, ..., v_k$ is a walk from x to y of length $k - j + i < k$. This contradicts the fact that the walk was as short as possible.

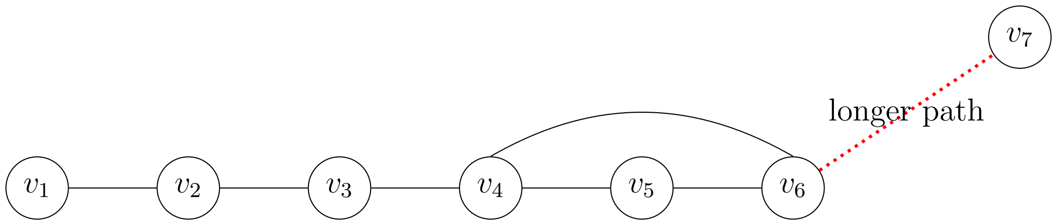
Proof 5.6 If there is a path from x to y and a path from y to z in a graph G, then there is a path from x to z in G.

Let $x = v_0, v_1, ..., v_k = y$ and $y = w_0, w_1, ...w_l = z$ be paths from x to y and y to z respectively. Now $x = v_0, v_1, ...v_k = y = w_0, w_1, ..., w_l = z$ is a walk from x to z. By what we proved above, there is a path from x to z.

Proof 5.7 If G is a graph and every vertex has degree at least 2, then G has a cycle.

Let $v_0, v_1, ..., v_k$ be a longest path in G. Since the path is longest, every number of v_k is in $\{v_0, v_1, ..., v_{k-1}\}$. Since $\deg(v_k) \geq 2$, there must be some $0 \leq i \leq k-2$ so that v_i is adjacent to v_k (if not then the path described is not the longest).

Now $v_i, v_{i+1}, ..., v_k, v_i$ is a cycle.



Theorem 5.4 *(Dirac 1952) If a graph G has $n \geq 3$ vertices and every vertex of G has degree $\geq \frac{n}{2}$, then G has a Hamilton cycle.*

Proof 5.8 Let $v_0, v_1, ..., v_k$ be a longest path of G.

Claim 1: There is a cycle of G whose vertices are $v_0, v_1, ..., v_k$ (in some order).

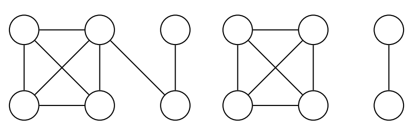
By maximality of the path, every neighbour of v_0 and every neighbour of v_k lies in the path. Since v_0 and v_k each have degree $\geq \frac{n}{2}$, we can find a neighbour v_l of v_k so that v_{l+1} is a neighbour of v_0. Then $v_0, v_1, ..., v_l, v_k, v_{k-1}, ..., v_{l+1}, v_0$ is a cycle.

Claim 2: Every vertex of G is in $\{v_0, ..., v_k\}$.

Since $\{v_0, v_1, ..., v_k\}$ contains v_0 and all its neighbours, $|\{v_0, ..., v_k\}| \geq \frac{n}{2} + 1$. If there is some $w \in V$ such that $w \notin \{v_0, ..., v_k\}$ then since $\deg(w) > \frac{n}{2}$, w has some neighbour in $\{v_0, ..., v_k\}$. But now $\{v_0, ..., v_k, w\}$ contains a path of length $k+1$, contradicting the maximality of the original path.

5.9 Connected

A graph G is **connected** if for all vertices x and y, G contains a walk (or path) from x to y.



Connected Disconnected

Proof 5.9 If x is a vertex of a graph G, and for all vertices y of G, there is a path from x to y, then G is connected. (Note that this is a weaker statement than our definition).

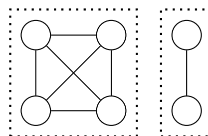
Let u, v be vertices of G. There is a walk from u to x and a walk from x to v, so there is a walk from u to v. Therefore, G is connected.

Which graphs are connected?

- Complete graphs
- Complete bipartite graphs are connected unless one side has no vertices (eg. $K_{0,3}$)

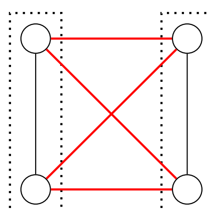
- Cycles
- Cubes

A **component** of a graph G is a maximal connected subgraph of G. That is, a connected subgraph H of G such that no connected subgraph H' of G has H as a proper subgraph.



5.10 Cut

Let (A, B) be a partition of the vertex set of a graph G ($A \cup B = V$ and $A \cap B = \varnothing$). The **cut** induced by (A, B) denotes the set of edges with one end in A and the other in B.



If the cut induced by (A, B) is the entire edge set, then (A, B) is a bipartition so the graph is bipartite. If $A, B \neq \varnothing$ but the cut induced by (A, B) is empty, then graph is disconnected.

Theorem 5.5 *Let G be a graph. G is connected if and only if there does not exist a partition (A, B) of V such that $A, B \neq \varnothing$ and the cut induced by (A, B) is empty*

Proof 5.10 Suppose that G is connected, but there exists a partition (A, B) of V inducing an empty cut with $A \neq \varnothing$ and $B \neq \varnothing$.

Let $u \in A, v \in B$. By connectedness, G contains a path $u = u_0, u_1, ..., u_k = v$. Note that $u_0 \in A, v_k \in B$. Let $0 \leq i < k$ be maximal such that $u_i \in A$. By maximality, $u_{i+1} \in B$, so G contains an edge from A to B. This is a contradiction.

Conversely, suppose that G is disconnected. Let C be a component of G. Let V_C be the set of vertices in C. Since C is connected and G is not, we know that $V_C \subsetneq V$ and $V_C \neq \varnothing$ so $(V_C, V \setminus V_C)$ is a partition of V into nonempty parts. Since C is a maximal connected subgraph, there is no edge from a vertex in V_C to one in $V \setminus V_C$.

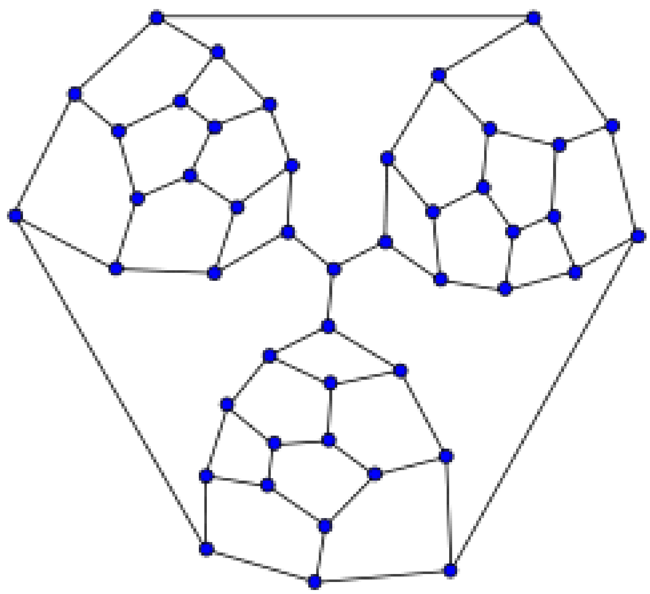
Theorem 5.6 *(Chvatal 1972)*

If G is a graph whose vertices have degrees $d_1 \leq d_2 \leq d_s \leq \ldots \leq d_n$ and for each $i \leq \frac{n}{2}$, either $d_i > i$ or $d_{n-i} \geq n-i$, then G is Hamiltonian.

For $k \in \mathbb{N}$, a graph is ***k*-connected** if for every pair of vertices u, v there are k internally disjoint paths from u to v.

Tait Conjecture: Every 3-connected graph that is planar is Hamiltonian.

The Tutte graph is a planar 3-connected graph but is not Hamiltonian.

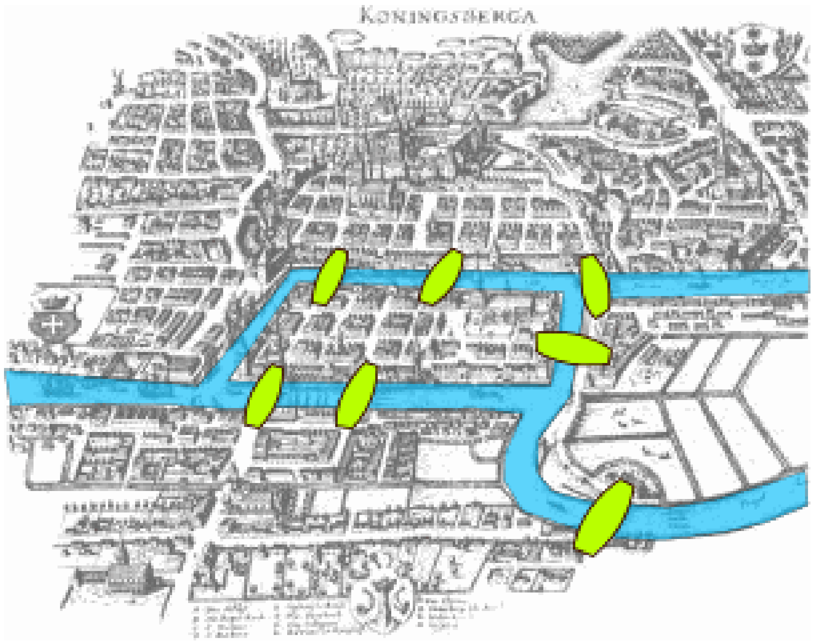


Theorem 5.7 *(Tutte)*

Every 4-connected planar graph is Hamiltonian.

5.11 Euler Tour

Inspired by the problem, "Can we walk around Konigsberg, crossing each bridge once, and returning to the start?"



An **Euler tour** in a graph is a closed walk containing each edge exactly once. A graph containing an Euler toupr is Eulerian.

Theorem 5.8 *If G has an Euler tour, then every vertex of G has even degrees.*

Proof 5.11 Let $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_k, v_k = v_0$ be an Euler tour.

Let v be a vertex of G. Each occurence of v in the sequence $v_0, v_1, ..., v_{k-1}$ has an edge both before and after it in the tour (where we consider e_k to be before v_0). Since the tour includes each edge exactly once, this means that every such v has even degree. $\square$

Theorem 5.9 *If G is a connected graph in which every vertex has even degree, then G has an Euler Tour.*

Proof 5.12 The theorem is trivial if there are no edges. Let $m > 0$ and suppose inducitvely that the result holds for all graphs on $< m$ edges.

Let G be a connected graph with m edges in which every vertex has an even degree. Let $v_0, e_1, v_1, v_2, ..., v_{k-1}, e_k, v_k = v_0$ be a closed walk of G with as many edges as possible. Let $F = \{e_1, e_2, ..., e_k\}$.

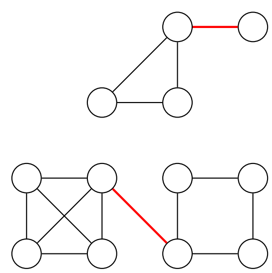
Since every vertex has even degree and G is connected, every vertex has degree ≥ 2 so G has a cycle (5.7). Therefore F contains at least as many edges as the cycle so $F \neq \varnothing$. If $F = E$, then the graph has an Euler Tour.

Then consider when $F \neq E$. Let $H = (V, E \setminus F)$ be the subraph of G formed by removing all edges in F. Since the subgraph (V, F) is Eulerian, every vertex is incident with an even number of edges in F, so removing F gives a graph in which every vertex has even degree. Note since $F \neq E$, that H has > 1 edge. Let C be a component of H that contains an edge.

Now C is connected, has $< m$ edges and every vertex has an even degree. So by the inductive hypothesis, C has an Euler Tour $w_0, f_1, w_1, f_2, ..., f_l, w_l = w_0$. Since G is connected, there is a vertex x of C incident with an edge in F. Now we can adjoin the walks $v_0, e_1, v_1, ..., e_k, v_k$ and $w_0, f_1, w_1, ..., w_l$ at their common vertex x to create a closed walk not repeating edges in G. Such a walk is longer than our original one which is a contradiction. □

5.12 Bridges

An edge e is a **bridge** of a graph G if the graph $G - e$ has more components than G. If $G = (V, E)$ then $G - e$ is the graph $(V, E \setminus \{e\})$.



Theorem 5.10 *$e = uv$ is a bridge of a graph G iff u and v are in difference components of $G - e$.*

Theorem 5.11 *An edge $e = uv$ is a bridge of a graph G iff it is not contained in a cycle of G.*

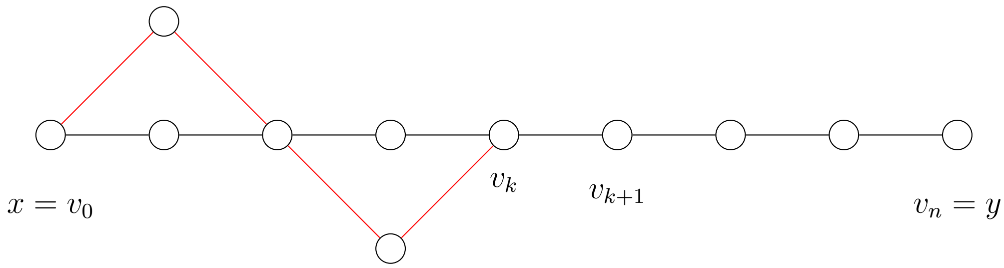
Proof 5.13 Suppose e is contained in a cycle C. Then the edges in $C - e$ form a path from u to v in $G - e$ so u and v are in the same component of $G - e$. Then by 5.10, e is not a bridge.

All of these implications work in reverse so we can prove the converse in a similar manner. □

Proof 5.14 If x and y are vertices of a connected graph G with no bridge, then G contains two edge-disjoint paths from x to y.

Let $x = v_0, v_1, ..., v_n = y$ be a path from x to y in G. Let $k \in \{0, 1, ..., n\}$ be maximal so that G contains two edge disjoin paths from x to v_k. If $v_k = v_n$, the theorem holds so suppose $k < n$. Let P, P' be edge-disjoint paths from x to v_k. The edge $v_k v_{k+1}$ is not a bridge so there is some path Q' from v_{k+1} to v_k that does not contain the edge $v_k v_{k+1}$.

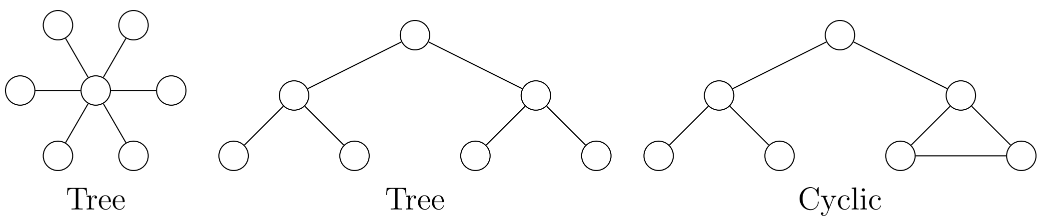
Let w be the first vertex of Q' that is contained in $P \cup P'$ and let Q be the subpath of Q' from v_{k+1} to w. Now the edges in P, P', Q and $\{v_k v_{k+1}\}$ contain edge-disjoin paths from x to v_{k+1} contradicting the maximality of k.



5.13 Trees

A **tree** is a connected graph with no cycles (acyclic graph).

A **leaf** of a tree is a degree-1 vertex.



Proof 5.15 A connected graph G is a tree iff every edge is a bridge.

We saw in 5.11 that an edge is a bridge iff it is contained in no cycle. This result follows. □

Proof 5.16 Every tree on ≥ 2 vertices has ≥ 2 leaves.

Let $v_0, v_1, ..., v_k$ be a longest path. By maximality, every neighbour of v_0 or v_k is in the path. By acylicity, v_0 and v_k have only neighbours v_1, v_{k-1} respectively. So $\deg(v_0) = \deg(v_k) = 1$. Then v_0, v_k are leaves. □

Proof 5.17 If T is a tree on n vertices, then T has $n - 1$ edges.

Trivial if $n = 1$. Suppose that the statement holds for every tree on k vertices for some $k > 1$. Let T be a tree on $k + 1$ vertices. Let v be a leaf of T and let T' be the graph obtained by removing v and a single incident edge from T (by 5.16).

T' is acyclic since T is acyclic. If x, y are vertices of T', then by connectedness of T, there is a path of T from x to y. Since $\deg(v) = 1$ this path does not contain v so it is also a path of T'. Therefore T' is connected and is a tree. T' has k vertices so it has $k - 1$ edges. Therefore T has k edges as required. □

Proof 5.18 Trees are bipartite.

Prove by removing a leaf and using induction. □

A **spanning tree** of a connected graph G is a subgraph of G that is a tree with the same vertex set as G.

Proof 5.19 Every connected graph has a spanning tree.

Let $G = (V, E)$ be a connected graph. Let F be a minimal subset of E so that the graph $H = (V, F)$ is connected.

Since F is minimal, the graph $H - e$ is disconnected for every $e \in F$, so every edge of H is a bridge. Thus H is a gree so it is a spanning tree. □

Proof 5.20 A graph G is bipartite iff it contains no odd cycle.

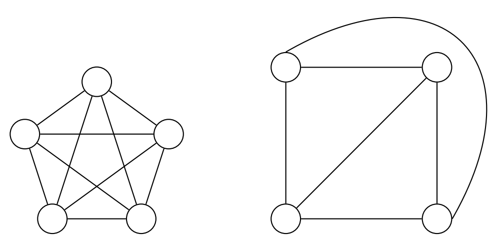
We may assume that G is connected. If not, then consider each component individually. Let T be a spanning tree of G. Suppose G has no odd cycles. We know trees are bipartite. Let (A, B) be a bipartition of T.

We'll show that (A, B) is also a bipartition of G. Suppose otherwise. Let x, y be adjacent vertices of G that are both in A or both in B. Let $x = u_0, ..., u_k = y$ be a path from x to y in T.

Since each edge of T has an end in A and an end in B, vertices in this path alternate between A and B. The ends are in the same set, so the length k is even. $x = u_0, u_1, ..., u_k = y = x$ is an odd cycle of G, a contradiction. We proved the converse earlier in Proof 5.2. □

5.14 Planar Graph

A **drawing** of a graph G is a subset of the plane such that every vertex corresponds to a distinct point, every edge corresponds to an open arc and the closure of each edge is exactly its endpoints.



K_5 is not planar   Planar embedding of K_4

We we draw any graph in the plane such that edges only meet at vertices?

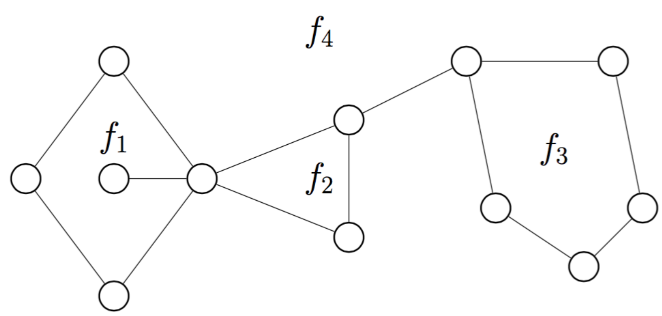
A graph G is **planar** if there is a drawing of G in the plane so that every vertex B is mapped to a distinct point and the intersections of the edges are disjoint. Such a drawing is called a **planar embedding** of G or a **planar map**.

Note if G is disconnected then G is planar iff every component of G is planar.

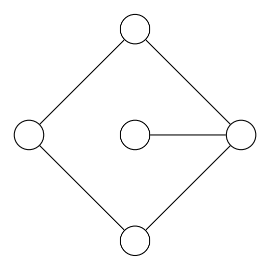
Theorem 5.12 *(Fary's Theorem) If G is planar then G can be embedded in the plane using only straight lines.*

If G is embedded in the plane P, the closures of the connected components of $P \setminus G$ are the **faces** of the embedding. The unbounded face of an embedding is called the **outer face**.

The subgraph of G formed by the vertices and edges in the bounding of F is the **boundary** of F.



G


Boundary of f_1

A vertex or edge of G in the boundary of F is **incident** with F. As we "walk" along the boundary of F we set a closed walk in G. Such a walk is the **boundary walk** of F denoted W_F.

The **degree** of F is the length of W_F (number of edges in W_F).

Any edge e appears twice in the set of boundary walks for faces of G since e is part of the boundary of two faces (could be the same face twice).

Given G embedded in the plane, the bridges of G are exactly the edges that appear twice in some face boundary walk.

Proof 5.21 All trees T are planar.

In any embedding of T in the plane, we have exactly one face. And any edge of T is contained in the boundary walk twice. So $\deg(F) = 2|E(T)| = 2|V(T)| - 2$.

Theorem 5.13 *(Handshake Theorem for Faces)*

If we have a planar embedding of a connected graph G with faces $F_1, F_2, ..., F_k$, then

$$\sum_{i=1}^{k} \deg(F_i) = 2|E(G)|$$

Theorem 5.14 *(Euler's Formula)*

Let G be a connected graph with v vertices and e edges. If G has an embedding in the plane with f faces, then

$$v - e + f = 2$$

Proof 5.22 For a connected graph G with v vertices, the minimum number of edges in G is $v - 1 = e$ when G is a tree. Any embedding of a tree in the plane has one face. Then

$$v - e + f = v - (v - 1) + 1 = 2$$

Suppose the claim is true for graphs on v vertices and $< e$ edges (with $e \geq v$). Since $e \geq v$, G is not a tree and there is some edge of G that is not a bridge.

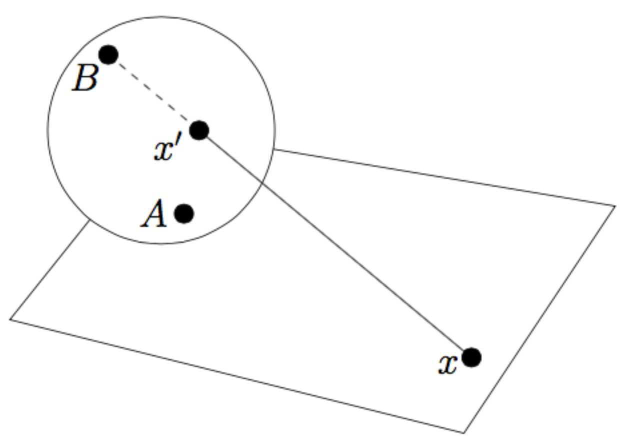
Suppose $\{a, b\}$ is a non-bridge edge of G. Consider $H = G \setminus \{a, b\}$. H has v vertices, $e - 1$ edges and H is connected. We showed earlier that an edge separates two faces and if the edge is not a bridge, then the two faces are different. Then by removing the edge, we join the two faces. So H has $f - 1$ faces.

Then by the inductive hypothesis

$$\begin{aligned} v - e + f &= 2 \\ v - (e - 1) + (f - 1) &= 2 \end{aligned}$$

as required. $\square$

5.14.1 Stereographic Projection



Any drawing on the plane can be converted to a drawing on a sphere via a stereographic projection. We'll have the sphere tangent to the plane at point A with point B antipodal to A on the sphere. Then any point x' on the sphere other than B can be mapped to a point x on the plane. If join B and x' with a line, we can have x be the intersection between the plane and the line.

Then a sphere minus a single point is equivalent to a plane. Our point B on the sphere cannot be mapped to a point on the plane and is a point on the plane at "infinite distance".

Theorem 5.15 *A graph is planar if and only if it can be drawn on a sphere.*

5.14.2 Platonic Graphs

A **fullerene** is a planar 3-regular graph with an embedding containing only degree 5 or 6 faces.

Proof 5.23 All fullerenes have exactly 12 degree 5 faces.

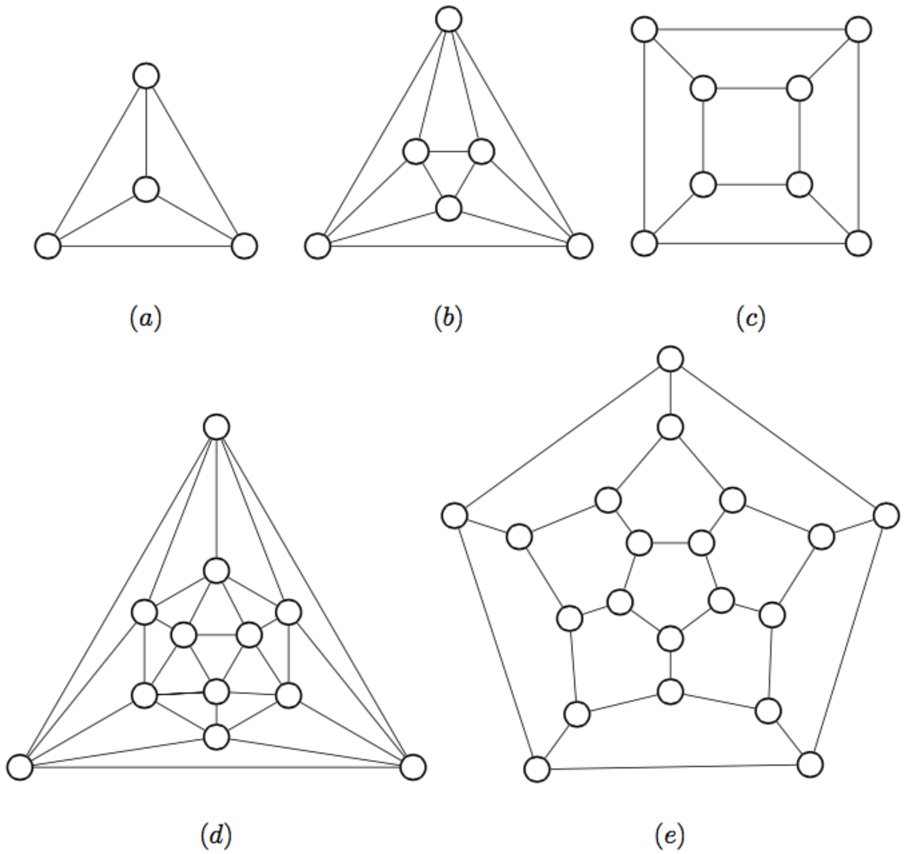
Let f_5 be the number of degree 5 faces and f_6 be the number of degree 6 faces. Then $f = f_5 + f_6$ by the definition of a fullerene.

By Euler's formula, $v - e + f_5 + f_6 = 2$. Theorem 5.13 gives $5f_5 + 6f_6 = 2e$. Then since a fullerene is 3-regular and by the Handshake Theorem we have $v - \frac{3}{2}v + f_5 + f_6 = 2$ and $5f_5 + 6f_6 = 3v$. Rearranging and equating f_6 in each equation gives

$$\begin{aligned} \frac{3v - 5f_5}{6} &= 2 + \frac{1}{2}v - f_5 \\ f_5 &= 12 \end{aligned}$$

□

A graph is **platonic** if it is d-regular (with $d \geq 3$) and has an embedding in the plane where all faces have degree d^* with $d^* \geq 3$.



The 5 platonic graphs.

Theorem 5.16 *There are exactly 5 platonic graphs.*

Proof 5.24 A platonic graph G has $(d, d^*) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$. Since G is d-regular and all faces have degree d^*,

$$\begin{aligned} dv &= 2e \\ v &= \frac{2e}{d} \\ d^* f &= 2e \\ f &= \frac{2e}{d^*} \end{aligned}$$

By Euler's formula,

$$\begin{aligned} \frac{2e}{d} - e + \frac{2e}{d^*} &= 2 \\ \frac{2}{d} + \frac{2}{d^*} &= \frac{2}{e} + 1 \end{aligned}$$

For any e, $1 + \frac{2}{e} > 1$.

Then if $d \geq 4$ and $d^* \geq 4$, then $\frac{2}{d} + \frac{2}{d^*} \leq 1$. If $d = 3$ and $d^* \geq 6$, then $\frac{2}{d} + \frac{2}{d^*} \leq 1$. This is a contradiction. $\square$

Proof 5.25 We'll prove that there are 5 platonic graphs.

If G is platonic with vertex degree d and face degree d^*,

$$e = \frac{2dd^*}{2d + 2d^* - dd^*}$$

This can be shown using Euler's formula and that $v = \frac{2e}{d}$ and $f = \frac{2e}{d^*}$.

So for each (d, d^*) we have v, e, f as determined. Each tuple gives one platonic graph.

Proof 5.26 If G is connected and not a tree, then the boundary of every face in a planar embedding of G contains a cycle.

Since G has a cycle, it has more than one face. Therefore, every face f is adjacent to at least one other face g.

Let $e = v_0 v_1$ be an edge that is incident with both f and g. Let H be the component in the boundary graph of face f containing the edge e_1. Let

$$W_f = (v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_{n-1}, v_0)$$

be the boundary walk of f. Since the edge e_1 is incident with both f and g, it is contained in W_f precisely once.

The edge e_1 is not a bridge of H because $(v_1, e_2, v_2, ..., v_{n-1}, e_n, v_0)$ is a walk from v_1 to v_0 in $H - e_1$. Therefore H contains a cycle. $\square$

To prove a graph is non-planar, we usually prove a property true for all planar graphs and then show that a graph does not have this property.

Proof 5.27 If G is a connected planar graph with $p \geq 3$ vertices and q edges, then $q \leq 3p - 6$.

If G is a tree, then the statement holds because $q = p - 1$. If G is not a tree, consider a planar embedding of G with p vertices, q edges and r faces. By the Handshake theorem for faces,

$$2q = \sum_{f \in F} \deg(f)$$

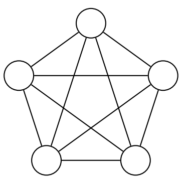
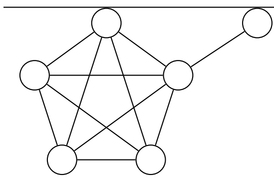
. Each face has degree ≥ 3 since the boundary of every face contains a cycle. Then

$$2q \geq 3r$$
$$r \leq \frac{2}{3} q$$

By Euler's Formula,

$$
\begin{aligned}
2 &= p - q + r \\
2 &\leq p - q + \frac{2}{3}q \\
2 &\leq p - \frac{1}{3}q \\
q &\leq 3p - 6
\end{aligned}
$$

$\square$

	K_5 has 10 edges and 5 vertices. $10 > 9$ so it cannot be planar.
	This has 11 edges and 6 vertices. $11 \leq 12$ so it does not fail our test (but we know it is non-planar since K_5 is non-planar).

Proof 5.28 If G is a connected planar graph that is not a tree with p vertices, q edges and every cycle has length $\geq d$, then $q \leq \frac{d}{d-2}(p-2)$.

Since every face boundary contains a cycle, $\deg(f) \geq d$ for each face f. By handshaking, $2q = \sum \deg(f) \geq dr$ so $r \leq \frac{2}{d}q$.

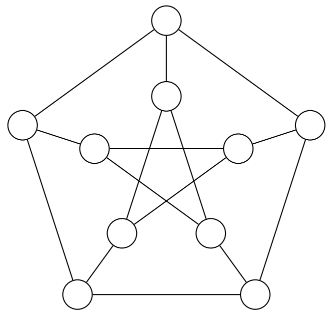
By Euler's formula

$$
\begin{aligned}
2 &= p - q + r \\
2 &\leq p - q + \frac{2}{d}q \\
q(1 - \frac{2}{d}) &\leq p - 2
\end{aligned}
$$

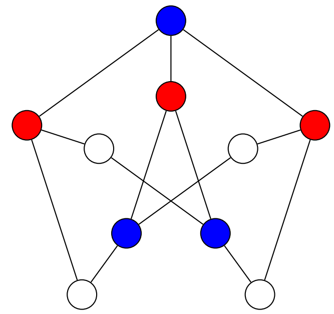
So $q \leq \frac{d}{d-2}(p-2)$ $\square$

Then $K_{3,3}$ is non-planar since every cycle has length at least 4, it has 9 edges and 6 vertices.

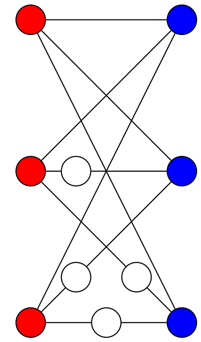
Is the Petersen graph planar? We can remove some edges from it. Notice that the graph below is homeomorphic to $K_{3,3}$, which is non-planar. Then the Petersen graph is non planar since it contains $K_{3,3}$ which is non-planar.



Petersen Graph



Petersen Graph with edges removed



Subdivision of $K_{3,3}$

A **subdivision** of a graph G is a graph obtained by replacing each edge of G by a path of length ≥ 1.

Theorem 5.17 *If H is a subdivision of a graph G, then H is planar iff G is planar.*

As a corollary, if H is a nonplanar graph and G is a graph containing a subdivision of H as a subgraph, then G is nonplanar.

Theorem 5.18 *(Kuratowski's Theorem)*

G is planar iff G contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.

5.15 Graph Coloring

Let $k \in \mathbb{N}$. A ***k*-coloring** of a graph $G = (V, E)$ is a function from V to a set of size k (whose elements are called **colors**) so that adjacent vertices are mapped to different colors always.

A graph with a k-coloring is ***k*-colorable**.

G is bipartite iff G is 2-colorable. The complete graph K_n is n-colorable but not $(n-1)$-colorable. The cycle C_n is 2-colorable iff n is even and is 3-colorable if n is odd.

Theorem 5.19 *(Four Colour Theorem)*

Every planar graph is 4-colourable.

The proof for the Four Colour Theorem is hard to prove. We'll prove the six-colour theorem instead by first proving the following lemma.

Proof 5.29 Every planar graph has a vertex of degree ≤ 5.

Let $G = (V, E)$ be a planar graph. We know that $|E| \leq 3|V| - 6$ by Proof 5.27. The handshake theorem shows that

$$\sum_{v \in V} \deg(v) = 2|E|$$

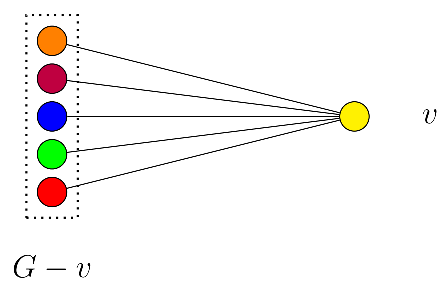
$$\frac{\sum_{v \in V} \deg(v)}{|V|} = \frac{2|E|}{|V|}$$

$$\frac{\sum_{v \in V} \deg(v)}{|V|} \leq \frac{2(3|V| - 6)}{|V|} \frac{\sum_{v \in V} \deg(v)}{|V|} \qquad \leq 6 - \frac{12}{|V|}$$

Then the average degree is $\leq 6 - \frac{12}{|V|} < 6$ so G has a vertex of degree ≤ 5. $\square$

Proof 5.30 Prove the six-colour theorem by induction on number of vertices. If G has ≤ 6 vertices, it is trivial. Suppose for $n \geq 6$, the theorem holds for every planar graph on n vertices. Let G' be a planar graph on $n + 1$ vertices.

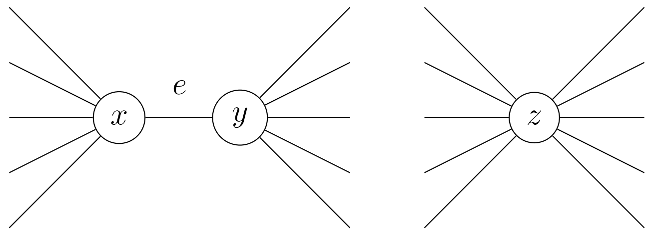
Let v be a vertex of degree ≤ 5 by Proof 5.29. Inductively, $G - v$ has a 6-colouring. Some colour is not used by any neighbour of v since it has less than 5 adjacent vertices. Assigning this colour to v gives a 6-colouring of G. $\square$



There exists a v in a planar graph with degree ≤ 5.

5.15.1 Contraction

If $e = xy$ is an edge of a graph $G = (V, E)$ then G/e denotes the graph with vertex set $(V \setminus \{x, y\}) \cup \{z\}$ where z is a new vertex not in V and edge set $\{uv : uv \in E$ and $\{u, v\} \cap \{x, y\} = \varnothing\} \cup \{wz : wx \in E$ or $wy \in E, w \notin \{x, y\}\}$.



G $\qquad$ G/e

Note that the contraction of a planar graph is also planar.

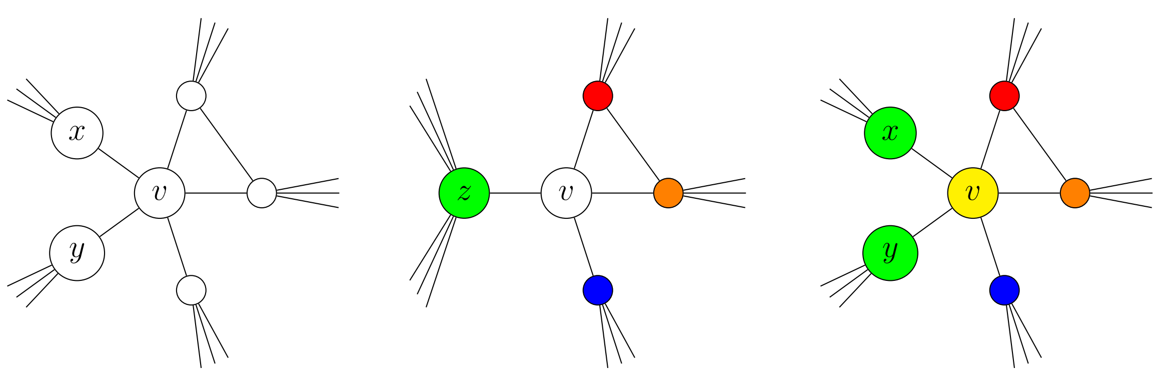
Theorem 5.20 *Every planar graph is 5-colourable.*

Proof 5.31 We'll prove by induction on number of vertices. It is trivial if $|V| \leq 5$. Suppose the result is true for every graph on $\leq n$ vertices where $n \geq 5$. Let G be a graph on $n+1$ vertices. Let v be a vertex of G of degree ≤ 5.

Consider the case when $\deg(v) \leq 4$. Inductively, $G - v$ has a 5-colouring. Some colour is not used by any neighbour of v in this colouring. Assigning that colour to v gives a 5-colouring of G.

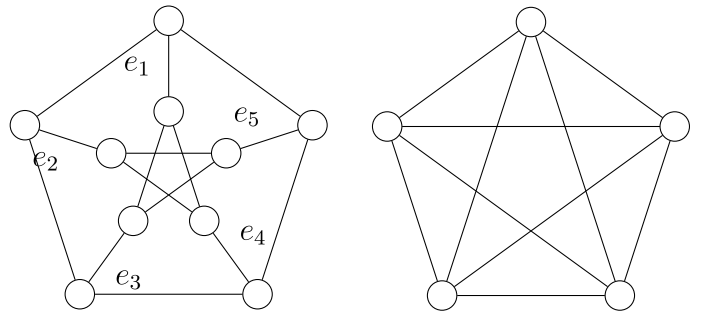
Now consider when $\deg(v) = 5$. G has no K_5 subgraph since it is planar. Then there are neighbours x, y of v that are nonadjacent in G. Let $e = xv, f = yv, H = G/e/f$. H is planar and less vertices than G so it has a 5-colouring. Suppose that the vertex z to which x, y, v are identified is assigned colour c in this colouring of H.

Now assigning c to x and y and colouring every vertex in $V \setminus \{x, y, v\}$ according to the colour it receives in H gives a colouring of $G - v$ in which x and y both have the same colour. Now the neighbours of v use ≤ 4 colours in this colouring of $G - v$ so we extending it to a 5-colouring of G as before. $\square$



G $\qquad$ Colouring of H $\qquad$ Colouring of G

We can show the Petersen graph is nonplanar using contractions.



P $\qquad$ $P/\{e_1, e_2, e_3, e_4, e_5\} = K_5$

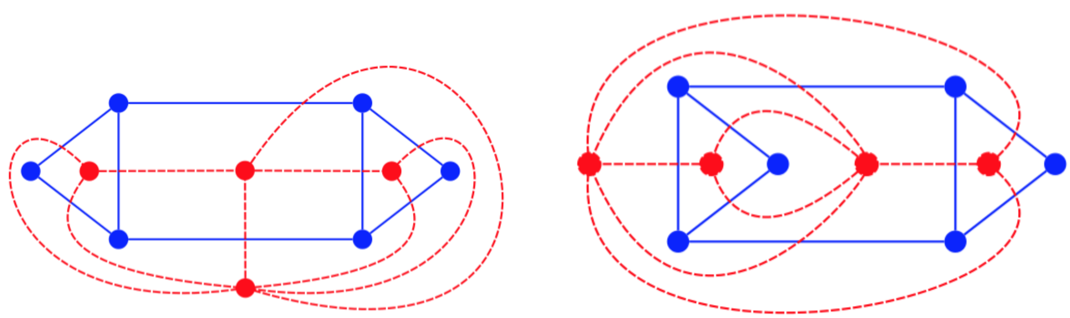
The **deletion** of an edge e in a graph G written $G \setminus e$ is the graph obtained by removing e from G. It can also be written $G - e$.

Theorem 5.21 *Kuratowski's Theorem (Minor Version)*

G is planar iff neither K_5 nor $K_{3,3}$ can be obtained from G by contracting/deleting edges and removing vertices.

5.15.2 Planar Dual

Let G be a *connected* planar *embedding* of a graph. The **planar dual** of G is the graph G^* such that the set of vertices of G^* is the set of faces of G and two vertices of G^* are joined by an edge iff the corresponding faces are adjacent in G.



1. G^* has a drawing on top of G so that each edge of G^* crosses exactly one edge of G and each vertex of G^* is drawn inside its corresponding face.
2. Each edge of G^* corresponds naturally to a unique edge of G. In particular, G and G^* have the same number of edges.
3. The faces of G^* correspond naturally to vertices of G.
4. $G^* * = G$ if G (requires connectedness of G)

5. $(G/e)^* = G * \backslash e$ and $(G \setminus e)^* = G^*/e$

6. G^* may have multiple edges or loops when G does not.

7. Different embeddings of G may have nonisomorphic duals (see graphs above)

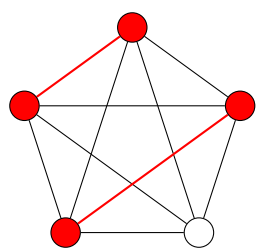
8. Platonic graphs come in dual pairs.

5.16 Matchings and Covers

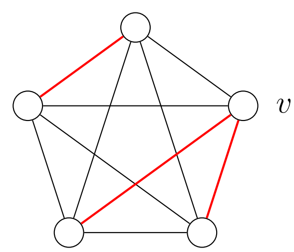
Given a graph $G = (V, E)$, a **matching** of G is a set $M \subseteq E$ so that each vertex of G is incident with at most one edge in M.

A vertex incident with an edge of M is **saturated**. If a vertex is not incident with an edge it is **unsaturated**.

If every vertex is saturated, the M is a **perfect matching**.



Edges in the matching and saturated vertices are in red



Not a matching since there are two incident edges to v

If M is a matching of a graph G, a path $v_0, v_1, ..., v_k$ of G is an ***M*-alternating path** if either $v_i v_{i+1} \in M$ iff i is even or $v_i v_{i+1} \in M$ iff i is odd.

If $v_0 v_1 \notin M$ and $v_{k-1} v_k \notin M$ and v_0, v_k are unsaturated then the M-alternating path is an **augmenting path**. Note that every augmenting path has odd length.

Proof 5.32 If M is a matching of a graph G and M has an augmented path, then M is not a maximum matching of G.

If $v_0, v_1, ..., v_k$ is an augmenting path, then $(M \backslash \{v_1 v_2, v_3 v_4, ..., v_{k-2} v_{k-1}\}) \cup \{v_0 v_1, v_2 v_3, ..., v_{k-1} v_k\}$ is a matching of G of size $|M| + 1$. So M is not a maximal matching. $\square$

A **cover** of a graph $G = (V, E)$ is a set $C \subseteq V$ so that every edge of G is incident with a vertex in C.

Note that the vertex set V is trivially a cover. In a bipartite graph with bipartition (A, B), both A and B are covers.

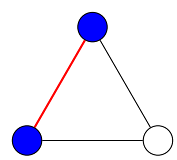
Proof 5.33 If M is a matching of G and C is a cover of G, then $|M| \leq |C|$.

Since C is a cover, it contains at least one end from each edge in M. The ends of these edges are all distinct so $|C| \geq |M|$. $\square$

Proof 5.34 If M is a matching and C is a cover of G, and $|M| = |C|$ then M is a maximal matching and C is a minimal cover.

By the previous proof, every matching M' has size $|M'| \leq |C| = |M|$ so M is a maximal matching. Similarly, every cover C' has size $|C'| \geq |M| = |C|$ so C' is a minimal cover. $\square$

Note that there exists graph G such that $|M| \neq |C|$ for a maximal matching M of G and minimal cover C of G.



Maximal matching in red and minimal cover in blue.

Let $v(G)$ denote the size of a maximal matching of G and $\tau(G)$ denote the size of a minimal cover.

Theorem 5.22 *(Konig's Theorem)*

In a bipartite graph, the maximum size of a matching is equal to the minimum size of a cover.

Proof 5.35 Of Konig's Theorem

The X-Y Construction: Let G be a bipartite graph with bipartition (A, B). Let M be a matching of G.

Let X_0 be the set of unsaturated vertices in A. Let Z be the set of all vertices v of G so that there is an alternating path from some $x \in X_0$ to v. Let $X = Z \cap A, Y = Z \cap B$.

For each $v \in Z$, let $P(v)$ be an alternating path from som $x \in X_0$ to v. Note that since G is bipartite and all vertices in X_0 are in A,

1. If $v \in X$ then $P(v)$ has even length and its last edge is in M since $v \in A$
2. If $v \in Y$ then $P(v)$ has odd length and its last edge is not in M since $v \in B$.

Lemma: Given G, A, B, X, Y as above

a) There is no edge of G from X to $B \setminus Y$.

b) $C = (A \setminus X) \cup Y$ is a cover of G.

c) There is no edge in M from $A \setminus X$ to Y.

d) Let Y_0 be the set of unsaturated vertices in Y. Then $|M| = |C| - |Y_0|$.

e) For every $y \in Y_0$, $P(y)$ is an augmenting path.

Lemma A: If xv is an edge with $x \in X, v \in B \setminus Y$ then $P(x), x$ is an alternating path from some vertex in X_0 to v, contradicting $v \notin Y$.

Lemma B: Follows from Lemma A and the definition of a cover.

Lemma C: If yv is an edge in M and $y \in Y, v \in A \setminus X$ then $P(y), v$ is an alternating path from some vertex in X_0 to v, contradicting $v \notin X$.

Lemma D: By Lemma A and C, every edge in M is either from X to Y or from $A \setminus X$ to $B \setminus Y$. There are $|Y| - |Y_0|$ edges of the first type and since every vertex in $A \setminus X$ is saturated, there are $|A \setminus X|$ edges of the second type. So the size of $|M| = |Y| - |Y_0| + |A| \setminus X| = |C| - |Y_0|$.

Lemma E: Follows because both ends are unsaturated by definition.

Proof of Konig's Theorem: Let G be a bipartite graph with bipartition (A, B) and let M be a max matching of G. Construct X, X_0, Y, Y_0 as above. Since M is maximum, it has no augmenting paths. So by Lemma E, $Y_0 = \varnothing$. Then $C = (A \setminus X) \cup Y$ is a cover and by by Lemma D, $|M| = |C|$. So M is a max matching and C is a min cover. $\square$

Max Bipartite Matching Algorithm

Input: Bipartite graph G with bipartition (A, B)

Step 1: Let M be any matching of G (eg. $\varnothing$)

Step 2: Let $\hat{X}$ be the set of unsaturated vertices in A and $\hat{Y} = \varnothing$

Step 2a: (Grow $\hat{Y}$) For each vertex $v \in B \setminus \hat{Y}$ that is adjacent to a vertex $u \in \hat{X}$, add v to $\hat{Y}$ and let $\mathrm{pr}(v) = u$. (pr stands for parent)

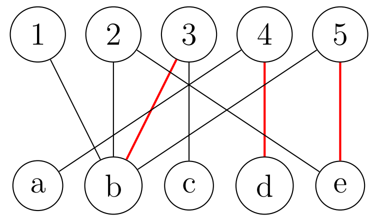
Step 2b: If $\hat{Y}$ contains an unsaturated vertex y, then y, $\mathrm{pr}(y)$, $\mathrm{pr}(\mathrm{pr}(y))$,... is an augmenting path. Use this path to make M bigger and repeat from 1.

Step 2c: If Step 2 added no new vertex to $\hat{Y}$, then M is a max matching and $C = (A \setminus \hat{X}) \cup \hat{Y}$. Return.

Step 3: (Grow $\hat{X}$) For each vertex $u \in A \setminus \hat{X}$ that is joined by an edge of M to a vertex $v \in \hat{Y}$, add u to $\hat{X}$, set $\mathrm{pr}(u)=v$. Goto 2.

Example 5.6 Example of algorithm.

1. Start with $\hat{X} = \{1, 2\}$, $\hat{Y} = \varnothing$.



2. Grow $\hat{Y}$:

   $\hat{X} = \{1, 2\}, \hat{Y} = \{b, e\}$, $\text{pr}(b) = 1$ and $\text{pr}(e) = 2$.

   2b and 2c do not apply so continue to step 3.

3. Grow $\hat{X}$:

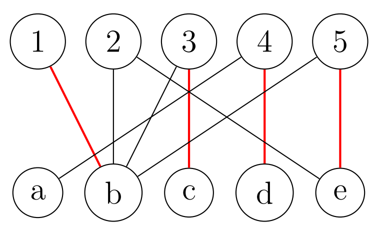
   $\hat{X} = \{1, 2, 3, 5\}, \hat{Y} = \{b, e\}$, $\text{pr}(3) = b$ and $\text{pr}(5) = e$

4. Grow $\hat{Y}$:

   $\hat{X} = \{1, 2, 3, 5\}, \hat{Y} = \{b, e, c, d\}$, $\text{pr}(c) = 3$ and $\text{pr}(d) = 3$.

   c is unsaturated and $c \in \hat{Y}$ so c, $\text{pr}(c)$, $\text{pr}(\text{pr}(c)), ... = c, 3, b, 1$ is augmenting.

5. We use the path above to grow M. Set $\hat{X} = \{2\}$ and $\hat{Y} = \varnothing$.



6. Grow $\hat{Y}$:

   $\hat{X} = \{2\}, \hat{Y} = \{b, e\}$, $\text{pr}(b) = 2$ and $\text{pr}(e) = 2$.

   2b and 2c do not apply.

7. Grow $\hat{X}$:

   $\hat{X} = \{2, 1, 5\}, \hat{Y} = \{b, e\}$, $\text{pr}(1) = b$ and $\text{pr}(5) = e$.

8. Grow $\hat{Y}$:

   $\hat{X} = \{2, 1, 5\}, \hat{Y} = \{b, e\}$.

   2c applies so M is a max matching. Then $A(\backslash\hat{X}) \cup \hat{Y} = \{3, 4, b, e\}$ is a min cover.

IF X is a set of vertices in a graph G, then the **neighbourhood** of X, denoted $N(X)$ is the set of vertices of G that are adjacent to a vertex in X.

Observe that if X is a set of vertices in a graph G with $|N(X)| < |X|$ then G has no matching saturating every vertex in X.

Theorem 5.23 *(Hall's Theorem)*

Let G be a bipartite graph with bipartition (A, B). Then G has a matching saturating A iff $|N(A')| \geq |A'|$ for all $A \subseteq A$.

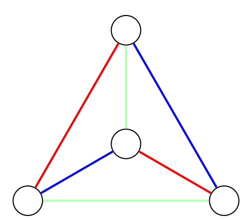
Proof 5.36 Clearly if there exists $A' \subseteq A$ with $|N(A')| < |A'|$ then G has no matching saturating A.

Conversely suppose that $|N(A')| \geq |A'|$ for all $A' \subseteq A$. To show that there is a matching saturating A, it suffices to show that A is a min cover (by Konig's Theorem).

Let C be a cover of G. Consider $A \setminus C$. Because C is a cover, every neighbour of a vertex in $A \setminus C$ is in $B \cap C$. So $N(A \setminus C) \subseteq B \cap C$. Therefore $|B \cap C| \geq |N(A \setminus C)| \geq |A \setminus C|$ by assumption. So $|C| = |C \cap A| + |C \cap B| = |C \cap A| + |A \setminus C| = |A|$. So A is a min cover.

Edge Colouring

A ***k*-edge colouring** of graph G is an assignment of a colour from a set of k colours to each edge of G so that edges sharing an end get different colours.



3-edge colouring of K_4

Theorem 5.24 *For $k > 0$, every k-regular bipartite graph has a perfect matching.*

Proof 5.37 Since the number of edges is $k|A| = k|B|$ we have $|A| = |B|$.

By Hall's theorem, for a perfect matching to exist we have to show that $|N(A')| \geq |A'|$ for all $A' \subseteq A$. Let $A' \subseteq A$. Let F be the set of edges from A' to $N(A')$.

Every edge with one end in A' is in F so $|F| = k|A'|$. Also every edge of F has one end in $N(A')$, so $|F| \leq k|N(A')|$. Then $k|A'| = |F| \leq k|N(A')|$ so $|A'| \leq |N(A')|$. Hall's theorem gives us the result. $\square$