

# MATH 239 NOTES

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From lectures by Peter Nelson

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# 1 Some Concepts

## 1.1 Binomial Theorem

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

## 1.2 Product of Polynomial

$$\begin{aligned} A(x)B(x) &= \left( \sum_{i \geq 0} a_i x^i \right) \left( \sum_{j \geq 0} b_j x^j \right) \\ &= \sum_{i \geq 0} \sum_{j \geq 0} a_i b_j x^{i+j}, \quad \text{now let } k = i \text{ and } n = i + j \\ &= \sum_{n \geq 0} \left( \sum_{k \geq 0}^n a_k b_{n-k} \right) x^n \end{aligned}$$

Or equivalently

$$[x^n]A(x)B(x) = \sum_{k \geq 0}^n a_k b_{n-k}$$

## 1.3 Sum Lemma

If  $S$  is a set with weight function  $w$  and  $A, B$  are sets so that  $A \cap B = \emptyset$ ,  $A \cup B = S$ , then  $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$ .

## 1.4 Product Lemma

If  $A, B$  be sets with weight function  $\alpha, \beta$  respectively. Then  $\Phi_A(x)\Phi_B(x) = \Phi_S(x)$  where  $S = A \times B$  and  $w(a, b) = \alpha(a) + \beta(b)$  is the weight function on  $S$ .

## 1.5 Negative Binomial Theorem

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$$

equivalently

$$[x^n](1-x)^{-k} = \binom{n+k-1}{k-1}$$

## 2 Counting Combinations

### 2.1 Intro using Fruit

In how many ways can you eat  $n$  pieces of fruit given that you must eat

- at most 5 apples
- at least 3 bananas
- an even number of cherries

The answer is  $[x^n] \underbrace{(1 + x + x^2 + x^3 + x^4 + x^5)}_{\text{apples}} \underbrace{(x^3 + x^4 + x^5 + \dots)}_{\text{bananas}} \underbrace{(1 + x^2 + x^4 + \dots)}_{\text{cherries}}$ .

$$\begin{aligned} &= [x^n] \left( \frac{1 - x^6}{1 - x} \right) \left( \frac{x^3}{1 - x} \right) \left( \frac{1}{1 - x^2} \right) \\ &= [x^n] \frac{x^3(1 - x^6)}{(1 - x)^3(1 + x)} \end{aligned}$$

Counting problems involving multiple selections can be encoded as coefficients. We'll now make this formal.

### 2.2 Sum Lemma

If  $S$  is a set with weight function  $w$  and  $A, B$  are sets so that  $A \cap B = \emptyset$ ,  $A \cup B = S$ , then  $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$ .

### 2.3 Product Lemma

If  $A, B$  be sets with weight function  $\alpha, \beta$  respectively. Then  $\Phi_A(x)\Phi_B(x) = \Phi_S(x)$  where  $S = A \times B$  and  $w(a, b) = \alpha(a) + \beta(b)$  is the weight function on  $S$ .

#### 2.3.1 Example Proving Binomial Theorem

Let  $S = \{\text{subsets of } [n]\}$  and  $w(A) = |A|$  for  $A \in S$ . So

$$\begin{aligned} \Phi_S(x) &= \sum_{k \geq 0} (\# \text{ elements of } S \text{ of weight } k) x^k \\ &= \sum_{k \geq 0} \binom{n}{k} x^k \end{aligned}$$

We will show inductively that this is  $(1+x)^n$ .

Base case  $(1+x)^0 = \binom{0}{0}x^0$  is trivial. Suppose it is true for  $n-1$  with  $n \geq 1$ .

Let  $T = \{\text{elements of } S \text{ containing } n\} = \{Y \cup \{n\} : Y \subseteq [n-1]\}$  and

$R = \{\text{elements of } S \text{ not containing } n\} = \{Y : Y \subseteq [n-1]\}$ . Clearly  $T \cap R = \emptyset$ . So by the Sum Lemma,  $\Phi_S(x) = \Phi_R(x) + \Phi_T(x)$ .

$$\begin{aligned}\Phi_R(x) &= \sum_{Y \subseteq [n-1]} x^{|Y|} \\ &= \sum_{k \geq 0} \binom{n-1}{k} x^k \\ &= (1+x)^{n-1} \\ \Phi_T(x) &= \sum_{Y \subseteq [n-1]} x^{|Y \cup \{n\}|} \\ &= \sum_{Y \subseteq [n-1]} x^{|Y|+1} \\ &= x \sum_{Y \subseteq [n-1]} x^{|Y|} \\ &= x(1+x)^{n-1}\end{aligned}$$

So  $\Phi_S(x) = (1+x)^{n-1} + x(1+x)^{n-1} = (1+x)^n$ .

## 2.4 Example with Fruit

For  $\leq 5$  apples,  $\geq 3$  blueberries and even number of cherries,

$$\begin{aligned}A &= \{0, 1, 2, 3, 4, 5\} \\ B &= \{3, 4, 5, 6, \dots\} \\ C &= \{0, 2, 4, 6, \dots\} \\ \Phi_A(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 \\ &= \frac{1-x^6}{1-x} \\ \Phi_B(x) &= x^3 + x^4 + x^5 + x^6 + \dots \\ &= \frac{x^3}{1-x} \\ \Phi_C(x) &= 1 + x^2 + x^4 + x^6 + \dots \\ &= \frac{1}{1-x^2}\end{aligned}$$

So the Product Lemma gives  $\Phi_S(x) = \Phi_A(x)\Phi_B(x)\Phi_C(x)$  where  $S = A \times B \times C$  and  $w(a, b, c) = w(a) + w(b) + w(c)$ . Then the number of valid selections for  $n$  pieces of fruit is  $[x^n]\Phi_S(x)$ .

$$\begin{aligned} [x^n]\Phi_S(x) &= [x^n]\Phi_A(x)\Phi_B(x)\Phi_C(x) \\ &= [x^n]\frac{1-x^6}{1-x}\frac{x^3}{1-x}\frac{1}{1-x^2} \\ &= [x^n]\frac{x^3(1-x^6)}{(1-x)^3(1+x)} \\ &= [x^{n-3}]\frac{1-x^6}{(1-x)^3(1+x)} \end{aligned}$$

## 2.5 Example of Change for \$1

**Q:** How many ways to make change for \$1?

A change of \$1 is a selection  $(a, b, c, d) \in (\mathbb{N}_0)^4$  such that  $5a + 10b + 25c + 100d = 100$ . Let

$$\begin{aligned} w_1(a) &= 5a \\ w_2(b) &= 10b \\ w_3(c) &= 25c \\ w_4(d) &= 100d \end{aligned}$$

$$\Phi_{\mathbb{N}_0^4}^w(x) = \Phi_{\mathbb{N}_0}^{w_1}(x)\Phi_{\mathbb{N}_0}^{w_2}(x)\Phi_{\mathbb{N}_0}^{w_3}(x)\Phi_{\mathbb{N}_0}^{w_4}(x)$$

## 2.6 Negative Binomial Theorem

**Prop:**

$$(1-x)^{-k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n$$

equivalently

$$[x^n](1-x)^{-k} = \binom{n+k-1}{k-1}$$

**Proof:**

$$\begin{aligned} [x^n](1-x)^k &= [x^n] \left( \frac{1}{1-x} \right)^k \\ &= [x^n] \underbrace{(1+x+x^2+\dots)(1+x+x^2+\dots)\dots(1+x+x^2+\dots)}_{k \text{ times}} \end{aligned}$$

This coefficient is the number of solutions to  $a_1 + a_2 + \dots + a_k = n$  where  $a_i \in \mathbb{N}$ . We show this with the product lemma. We have  $\Phi_{\mathbb{N}_0}(x) = 1 + x + x^2 + x^3 + \dots$  with respect to the weight function  $w(a) = a$ .

$$(1 + x + x^2 + \dots)^k = (\Phi_{\mathbb{N}_0}(x))^k \\ = \Phi_S(x)$$

Where  $S = (\mathbb{N}_0)^k$  and  $w = (a_1, a_2, \dots, a_k) = a_1 + a_2 + \dots + a_k$ .

Let  $T = \{(a_1, a_2, \dots, a_k) \in \mathbb{N}_0^k \mid a_1 + a_2 + \dots + a_k = n\}$  and  $R = \{\text{Binary strings of length } n + k - 1 \text{ with exactly } k - 1 \text{ ones}\}$ .

We know  $|T| = [x^n](1 - x)^{-k}$  and  $|R| = \binom{n+k-1}{k-1}$ .

We define a bijection  $f : T \rightarrow R$  by

$$f(a_1, a_2, \dots, a_k) = \underbrace{0\dots0}_a 1 \underbrace{0\dots0}_a 1 \dots 1 \underbrace{0\dots0}_a$$

and it's inverse by

$$f(\underbrace{0\dots0}_a 1 \underbrace{0\dots0}_a 1 \dots 1 \underbrace{0\dots0}_a) = (b_1, b_2, \dots, b_k)$$

Clearly  $f$  and  $g$  are inverses so  $f$  is a bijection and  $|T| = |R|$ . □

We can use the negative binomial theorem to go between rational expressions and power series.

**eg.**

$$(1 + 2x^2)^{-5} = \sum_{n \geq 0} \binom{n+4}{4} (-2x^2)^n \\ = \sum_{n \geq 0} (-2)^n \binom{n+4}{4} x^{2n}$$

## 2.7 Compositions

The ideas in the negative binomial theorem proof allude to a new type of combinatorial object.

Let  $n \in \mathbb{N}_0, k \in \mathbb{N}_0$ . A **composition** of  $n$  into  $k$  parts is a  $k$ -tuple  $(a_1, a_2, \dots, a_k)$  such that  $a_1 + a_2 + \dots + a_k = n$  and  $a_i \in \mathbb{N}$ .

**Example.** The compositions of 5 into 3 parts are  $(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)$ . Note that order matters. Ignoring order, we have **partitions** which are much harder to work with.

**Prop.**

There are  $\binom{n-1}{k-1}$  compositions of  $n$  with  $k$  parts.



**Proof.**

Let  $S = \{\text{Compositions of } n \text{ into } k \text{ parts}\}$ ,  $T = \{\text{solutions to } a_1 + a_2 + \dots + a_n \text{ with } a_i \in \mathbb{N}_0\}$ .

$f(a_1, a_2, \dots, a_k) = (a_1 - 1, a_2 - 1, \dots, a_k - 1)$  gives a bijection from  $S$  to  $T$ . By the material in the proof earlier,

$$|T| = \binom{(n - k + k - 1)}{k - 1}$$

$$|T| = |S| = \binom{n - 1}{k - 1}$$

□

**Prop.**

The number of compositions of  $n$  into any number of parts is  $2^{n-1}$ .

**Proof.**

By previous proposition, the number is  $\sum_{k \geq 1} \binom{n-1}{k-1} = 2^{n-1}$  by the binomial theorem.

## 2.8 Restricted Compositions

Often we will need to compute the number of compositions of  $n$  with various restrictions on the number of parts, or their sizes. The sum/product lemmas do this.

### 2.8.1 Small Parts

How many compositions of  $n$  have each part equal to 1 or 2.

- With  $k$  parts?
- With any number of parts?

Let  $S = \{1, 2\}$  and  $w(\sigma) = \sigma$  for each  $\sigma \in S$ . Then  $\Phi_S(x) = x + x^2$ .

Consider  $[x^n] \Phi_S(x)^k$ . By the product lemma, it is equal to the number of  $k$ -tuples  $(a_1, a_2, \dots, a_k) \in S^k$  with  $a_1 + a_2 + \dots + a_k = n$ . So this is the number of compositions of  $n$  into  $k$  parts of size 1 or 2.

$$\begin{aligned} [x^n] \Phi_S(x)^k &= [x^n] (x + x^2)^k \\ &= [x^n] x^k (1 + x)^k \\ &= [x^{n-k}] (1 + x)^k \\ &= \binom{k}{n - k} \end{aligned}$$

So the number of compositions of  $n$  into  $k$  parts of size 1 or 2 is  $\binom{k}{n-k}$ . So the number of compositions of  $n$  into any number of parts of size 1 or 2 is  $\sum_{k \geq 0} \binom{k}{n-k}$ .

Alternatively, the number of compositions of  $n$  into any number of parts of size 1 or 2 is

$$\begin{aligned} \sum_{k \geq 0} [x^n](x + x^2)^k &= [x^n] \sum_{k \geq 0} (x + x^2)^k \\ &= [x^n] \frac{1}{1 - x - x^2} \\ &= n\text{th Fibonacci number} \end{aligned}$$

## 2.8.2 Odd Parts

How many compositions of  $n$  have each part odd?

Let  $S = \{1, 3, 5, 7, \dots\}$  and  $w(\sigma) = \sigma$  for each  $\sigma \in S$ .

$$\begin{aligned} \Phi_S(x) &= x^1 + x^3 + x^5 + x^7 + \dots \\ &= x(1 + x^2 + x^4 + \dots) \\ &= \frac{x}{1 - x^2} \end{aligned}$$

Then the number of compositions of  $n$  into  $k$  odd parts is  $[x^n]\Phi_S(x)^k$ . So the number of compositions of  $n$  into any number of odd parts is

$$\begin{aligned} \sum_{k \geq 0} [x^n]\Phi_S(x)^k &= [x^n] \sum_{k \geq 0} \Phi_S(x)^k \\ &= [x^n] \frac{1}{1 - \Phi_S(x)} \\ &= [x^n] \frac{1}{1 - \frac{x}{1 - x^2}} \\ &= [x^n] \frac{1 - x^2}{1 - x - x^2} \end{aligned}$$

Let  $\frac{1-x^2}{1-x-x^2} = a_0 + a_1x + a_2x^2 + \dots$

Solving  $(1 - x - x^2)(a_0 + a_1x + a_2x^2 + \dots) = 1 - x^2$  we get

$$\begin{aligned} a_0 &= 1 \\ a_1 - a_0 &= 0 \\ a_2 - a_1 - a_0 &= -1 \\ a_k - a_{k-1} - a_{k-2} &= 0, \quad k \geq 3 \end{aligned}$$

Then  $a_0 = 1, a_1 = 1, a_2 = 1$  and  $a_k = a_{k-1} + a_{k-2}$  for  $k \geq 3$ . So the number of compositions of  $n$  into odd parts is the  $(n - 1)$ th Fibonacci number.

### 2.8.3 Combinatorial Proof of Compositions of Size 1 and 2

Let  $A_n = \{\text{compositions of } n \text{ into parts of size 1 or 2}\}$ . We need  $|A_n| = |A_{n-1}| + |A_{n-2}|$ . Let  $A'_n = \{\text{compositions of } n \text{ into parts of size 1 or 2 with last part 1}\}$ . Let  $A''_n = \{\text{compositions of } n \text{ into parts of size 1 or 2 with last part 2}\}$ .

Let  $f_1 : A'_n \rightarrow A_{n-1}$  be defined by  $f_1(a_1, a_2, \dots, a_k) = (a_1, a_2, \dots, a_{k-1})$ . Its inverse is  $f_1^{-1} : A_{n-1} \rightarrow A'_n$  defined by  $f_1^{-1}(b_1, b_2, \dots, b_k) = (b_1, \dots, b_k, 1)$ . A similar bijection can be found between  $A''_n$  and  $A_{n-2}$ .

So since  $|A_n| = |A'_n| + |A''_n|$ ,  $|A_n| = |A_{n-1}| + |A_{n-2}|$ . □

### 2.8.4 Combinatorial Proof of Odd Sized Compositions

Let  $T_n = \{\text{compositions of } n \text{ into parts of odd size}\}$ . Clearly  $|T_1| = |T_2| = 1$ . To show that  $T_{n+1}$  is the  $n$ th Fibonacci number, it suffices to show that  $|T_n| = |T_{n-1}| + |T_{n-2}|$  for  $n \geq 3$ .

We do this by defining a bijection  $f$  between  $T_n$  and  $T_{n-1} \cup T_{n-2}$ .

$$\begin{aligned} T_2 &= \{(1, 1)\} \\ T_3 &= \{(1, 1, 1), (3)\} \\ T_4 &= \{(1, 1, 1, 1), (1, 3), (3, 1)\} \\ T_5 &= \{(1, 1, 1, 1, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (5)\} \end{aligned}$$

Let  $f : T_n \rightarrow T_{n-1} \cup T_{n-2}$  be defined by

$$f(a_1, a_2, \dots, a_k) = \begin{cases} (a_1, a_2, \dots, a_{k-1}) & a_k = 1 \\ (a_1, a_2, \dots, a_k - 2) & a_k \neq 1 \end{cases}$$

and  $g : T_{n-1} \cup T_{n-2} \rightarrow T_n$  be defined by

$$g(a_1, a_2, \dots, a_k) = \begin{cases} (a_1, a_2, \dots, a_k, 1) & (a_1, \dots, a_k) \in T_{n-1} \\ (a_1, a_2, \dots, a_k + 2) & (a_1, \dots, a_k) \in T_{n-2} \end{cases}$$

Then  $g$  is the inverse of  $f$ . So  $f$  is a bijection and thus  $|T_n| = |T_{n-1} \cup T_{n-2}| = |T_{n-1}| + |T_{n-2}|$ . □

### 2.8.5 Relationship between Above Compositions

Let  $T_n = \{\text{compositions of } n \text{ into parts of odd size}\}$ . Let  $S_n = \{\text{compositions of } n \text{ into parts of size 1 or 2}\}$ . We'll show that  $|S_n| = |T_{n+1}|$  by finding a bijection.

We have  $(1, 3, 7, 5, 9, 3, 3, 1, 3) \in T_{35}$  can be mapped to

$$\left( \underbrace{1}_1, \underbrace{2, 1}_3, \underbrace{2, 2, 2, 1}_7, \underbrace{2, 2, 1}_5, \underbrace{2, 2, 2, 2, 1}_9, \underbrace{2, 1}_3, \underbrace{2, 1}_3, \underbrace{1}_1, \underbrace{2, 1}_3 \right)$$

This is done by transforming each element in the composition as a 1 prefixed by the appropriate number of 2s.

However, this results in compositions that always end in 1. So we remove the final 1 to map  $T_{35}$  to  $S_{34}$ . This rule can be formally defined as a bijection so  $|T_{n+1}| = |S_n|$ .  $\square$

### 3 Binary Strings

A binary string of length  $k$  is a  $k$ -tuple  $(a_1, \dots, a_k)$  where  $a_i \in \{0, 1\}$ . Equivalently, a member of  $\{0, 1\}^k$ . We usually suppress commas and brackets and write strings as  $a_1a_2\dots a_n$ .

If  $\sigma = s_1s_2\dots s_j$  and  $\tau = t_1t_2\dots t_k$  then  $\sigma\tau = s_1s_2\dots s_jt_1t_2\dots t_k$ . (**concatenation**)

We write  $l(\sigma)$  for the length of  $\sigma$ . So  $l(\sigma\tau) = l(\sigma) + l(\tau)$ .

$\sigma^k$  denotes  $\underbrace{\sigma\sigma\dots\sigma}_{k \text{ times}}$  and  $\sigma^0 = \epsilon$ .

If  $A, B$  are sets of strings then  $AB = \{\alpha\beta : \alpha \in A, \beta \in B\}$ .

We also define  $A^k = \underbrace{AAA\dots A}_{k \text{ times}}$ .

**Example 3.1**  $\{0, 1\}^7 = \{\text{strings of length 7}\}$

$$\begin{aligned} A^* &= \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup \dots \\ &= \bigcup_{k \geq 0} A^k \end{aligned}$$

A **substring** of  $s$  is a string  $b$  such that  $s = abc$  for some  $a, c$ .

A **block** of  $s$  is a maximal substring of  $s$  whose members are equal (ie. all 0 or 1).

#### 3.1 Ambiguity

If each such string in  $A^*$  can only be obtained from  $A^*$  in one way, then  $A^*$  is **unambiguous**. Other expressions can also be called ambiguous or unambiguous.

For example,  $\{0, 00\}\{0, 00, 000\}$  is ambiguous since 000 can be made in multiple ways.  $\{0, 1\}$  is unambiguous. Also for any set  $A$  such that  $\epsilon \in A$ ,  $A^*$  is ambiguous.

Is  $\{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^*$  ambiguous? No. It is unambiguous but generates all possible binary strings. We can decompose any string by taking all 1s in the front and 0s in the back into  $\{1\}^*$  and  $\{0\}^*$ .  $\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*$  captures blocks of 0s and 1s in the middle.

Another unambiguous expression generating all binary strings is  $\{0, 1\}^*$ . However, it is less useful than the previous expression for counting problems.

### 3.2 Strings and Generating Series

Let  $S$  be a set of binary strings with  $w(\sigma) = \text{length}(\sigma)$ . Then the number of strings of length  $n$  in  $S$  is  $[x^n]\Phi_S(x)$ .

**Theorem 3.1** *If  $S = A \cup B$  unambiguously, then  $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$ .*

*If  $S = AB$  unambiguously, then  $\Phi_S(x) = \Phi_A(x)\Phi_B(x)$ .*

*If  $S = A^*$  unambiguously, then  $\Phi_S(x) = \frac{1}{1-\Phi_S(x)}$ . Notice that  $\Phi_S(x)$  must have a zero constant term, which agrees with the fact that  $A$  is ambiguous if it contains  $\epsilon$ .*

**Example 3.2** Let  $S = \{\text{binary strings where each block of zero has even length}\}$ .

We know  $S = \{00, 1\}^*$  unambiguously. Then the number  $k$  of strings of length  $n$  in  $S$  is

$$\begin{aligned} k &= [x^n]\Phi_S(x) \\ &= [x^n]\frac{1}{1-\Phi_A(x)} \end{aligned}$$

where  $A = \{00, 1\}$ .  $\Phi_A(x) = x + x^2$  so  $[x^n]\Phi_S(x) = [x^n]\frac{1}{1-x-x^2}$ . Therefore the answer is the  $n$ th Fibonacci number.

**Example 3.3** Let  $S = \{\text{strings with exactly three blocks}\}$ .

We can decompose  $S$  as  $S = \underbrace{\{\{1\}\{1\}^*\{0\}\{0\}^*\{1\}\{1\}^*\}}_{A_1} \cup \underbrace{\{\{0\}\{0\}^*\{1\}\{1\}^*\{0\}\{0\}^*\}}_{A_0}$ . That

is,  $S = \{\text{strings of the form } 1\dots 10\dots 01\dots 1\} \cup \{\text{strings of the form } 0\dots 01\dots 10\dots 0\}$ .

$$\begin{aligned} \Phi_{A_1}(x) &= \Phi_{\{1\}}(x)\Phi_{\{1\}^*}(x)\Phi_{\{0\}}(x)\Phi_{\{0\}^*}(x)\Phi_{\{1\}}(x)\Phi_{\{1\}^*}(x) \\ &= (x) \left(\frac{1}{1-x}\right) (x) \left(\frac{1}{1-x}\right) (x) \left(\frac{1}{1-x}\right) \\ &= \frac{x^3}{(1-x)^3} \end{aligned}$$

Similarly  $\Phi_{A_0}(x) = \frac{x^3}{(1-x)^3}$ .

$$\begin{aligned}\Phi_S(x) &= \Phi_{A_0}(x) + \Phi_{A_1}(x) \\ &= \frac{2x^3}{(1-x)^3} \\ &= 2x^3 \sum_{n \geq 0} \binom{n+2}{2} x^n\end{aligned}$$

So the number of elements in  $S$  of length  $n$  is  $2 \binom{n-1}{2}$ .

This makes sense intuitively since we are picking two positions where the string swaps between repeating 0 and repeating 1. And the string can either start with 0 or 1.

**Example 3.4** Let  $S$  be the set of strings with all blocks with length  $\geq 2$ .

Then  $S = (\epsilon \cup \{00\}0^*)(\{11\}1^*\{00\}0^*)^*(\epsilon \cup \{11\}1^*)$ .

$$\begin{aligned}\Phi_S(x) &= \left(1 + \frac{x^2}{1-x}\right) \frac{1}{1 - \left(\frac{x^2}{1-x} \frac{x^2}{1-x}\right)} \left(1 + \frac{x^2}{1-x}\right) \\ &= \left(\frac{1-x+x^2}{1-x}\right)^2 \frac{(1-x)^2}{(1-x)^2 - x^4} \\ &= \frac{(1-x+x^2)^2}{(1-x)^2 - x^4} \\ &= \frac{1-x+x^2}{1-x-x^2}\end{aligned}$$

**Example 3.5** Let  $S$  be the set of strings where an even block of 0s cannot be followed by an odd number of block of 1s.

$$\begin{aligned}S &= 1^* \left( \underbrace{\{0(00)^*11^*\}}_{\text{odd 0s}} \cup \underbrace{\{00(00)^*11(11)^*\}}_{\text{even 0s}} \right)^* 0^* \\ \Phi_S(x) &= \frac{1}{1-x} \frac{1}{1 - \left(\frac{x}{1-x^2} \frac{x}{1-x} + \frac{x^2}{1-x^2} \frac{x^2}{1-x^2}\right)} \frac{1}{1-x} \\ &= \frac{(1+x)^2}{x(1+x^2+x^3)}\end{aligned}$$

**Example 3.6** Let  $S$  be the set of strings with no  $l$  consecutive 1s and no  $m$  consecutive 0s.

$$\begin{aligned}
S &= (0^* \setminus \{0^m 0^*\}) [(\{11^*\} \setminus \{1^l 1^*\}) (\{00^*\} \setminus \{0^m 0^*\})]^* (1^* \setminus \{1^l 1^*\}) \\
\Phi_S(x) &= \left( \frac{1}{1-x} - \frac{x^m}{1-x} \right) \left( \frac{1}{1 - \left( \frac{x}{1-x} - \frac{x^l}{1-x} \right) \left( \frac{x}{1-x} - \frac{x^m}{1-x} \right)} \right) \left( \frac{1}{1-x} - \frac{x^l}{1-x} \right) \\
&= \frac{1 - x^m - x^l + x^{m+l}}{1 - 2x + x^{m+1} + x^{l+1} - x^{m+l}} \quad (\text{after some algebra})
\end{aligned}$$

Considering  $l = 1, m = 1$ ,

$$\begin{aligned}
\Phi_S(x) &= \frac{1 - 2x + x^2}{1 - 2x + x^2 + x^2 - x^2} \\
&= 1
\end{aligned}$$

This makes sense since only  $\epsilon$  satisfies the constraints.

Considering  $l = 2, m = 2$ ,

$$\begin{aligned}
\Phi_S(x) &= \frac{1 - 2x + x^4}{1 - 2x + 2x^3 - x^4} \\
&= \frac{(1 - x^2)^2}{(1 - x^2)(1 - 2x + x^2)} \\
&= \frac{1 - x^2}{1 - 2x + x^2} \\
&= \frac{1 + x}{1 - x}
\end{aligned}$$

Then we have,

$$\begin{aligned}
1 + x &= a_0(1 - x) + a_1x(1 - x) + a_2x^2(1 - x) + \dots \\
a_0 &= 1 \\
-a_0 + a_1 &= 1 \Rightarrow a_1 = 2 \\
a_i - a_{i-1} &= 0 \Rightarrow a_{i+1} = a_i \forall i \geq 2
\end{aligned}$$

This makes sense since we can have either  $\epsilon$ , 0101...0101 or 1010...1010.

### 3.3 Recursive Decompositions

**Example 3.7** Let  $S$  be the set of all strings.

$S$  can be recursively described as  $S = \{\epsilon\} \cup S\{0, 1\}$ . We then have the generating function,

$$\begin{aligned}
\Phi_S(x) &= 1 + \Phi_S(x)(2x) \\
\Phi_S(x) - 2x\Phi_S(x) &= 1 \\
\Phi_S(x) &= \frac{1}{1-2x} \\
\Phi_S(x) &= \sum_{k \geq 0} 2^k x^k
\end{aligned}$$

Which gives us that there are  $2^k$  binary strings of length  $k$ , as expected.

**Example 3.8** Let  $S$  be the set of strings without 111.

$$\begin{aligned}
S &= \{\epsilon, 1, 11\} \cup S\{0, 01, 011\} \\
\Phi_S(x) &= (1 + x + x^2) + \Phi_S(x)(x + x^2 + x^3) \\
\Phi_S(x) &= \frac{1 + x + x^2}{1 - (x + x^2 + x^3)}
\end{aligned}$$

**Example 3.9** How many strings are there with no 11101?

Let  $L$  be the set of strings without 11101. Let  $M$  be the set of strings with 11101 at the end and not anywhere else in the string. Notice that  $L$  and  $M$  are disjoint.

$L \cup M = \{\epsilon\} \cup L\{0, 1\}$ . Adding a 0 or 1 won't add 11101 in the middle of the string but can add it to the end.

We need to find an expression for  $M$ . We don't have  $M = L\{11101\}$  since  $\{1110\}\{11101\}$  has two 11101 sequences.

$L\{11101\} = M \cup M\{1101\}$ . This accounts for the fact that we can create a second 11101 sequence by appending to  $M$ .

$$\begin{aligned}
\Phi_L(x) + \Phi_M(x) &= 1 + 2x\Phi_L(x) && \text{(from } L \cup M = \{\epsilon\} \cup L\{0, 1\}\text{)} \\
\Phi_L(x)x^5 &= \Phi_M(x) + \Phi_M(x)x^4 && \text{(from } L\{11101\} = M \cup M\{1101\}\text{)} \\
\Phi_M(x) &= \frac{x^5}{1+x^4}\Phi_L(x) \\
\Phi_L(x) + \frac{x^5}{1+x^4}\Phi_L(x) &= 1 + 2x\Phi_L(x) && \text{(substituting into first equation)} \\
\Phi_L(x) &= \frac{1}{1-2x + \frac{x^5}{1+x^4}} \\
\Phi_L(x) &= \frac{1-x^4}{1-2x-x^4+3x^5}
\end{aligned}$$



## 4 Evaluating Coefficients of Generating Series

### 4.1 Partial Fractions

**Example 4.1** Let  $f(x) = \frac{1+3x}{(1-x)(1+x)(1-2x)}$

$$\begin{aligned} f(x) &= \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-2x} \\ &= \frac{A(1+x)(1-2x) + B(1-x)(1-2x) + C(1-x)(1+x)}{(1-x)(1+x)(1-2x)} \end{aligned}$$

$$A + B + C = 1$$

$$-A - C = 3$$

$$-2A + 2B + C = 0$$

So we have  $A = -2, B = -\frac{1}{3}, C = \frac{1}{3}$ . Substituting,

$$\begin{aligned} f(x) &= \frac{-2}{1-x} - \frac{1}{3} \frac{1}{1+x} + \frac{10}{3} \frac{1}{1-2x} \\ &= -2 \sum_{k \geq 0} x^k - \frac{1}{3} \sum_{k \geq 0} (-x)^k + \frac{10}{3} \sum_{k \geq 0} (2x)^k \\ &= \sum_{k \geq 0} \left( -2 - \frac{1}{3}(-1)^k + \frac{10}{3}2^k \right) x^k \end{aligned}$$

Then  $[x^n]f(x) = -2 - \frac{1}{3}(-1)^n + \frac{10}{3}2^n$ .

**Theorem 4.1** Let  $f, g$  be polynomials with  $\deg(g) < \deg(f)$  and  $f$  has constant term 1. Then

$$\frac{g(x)}{f(x)} = \frac{h_1(x)}{(1 - \Theta_1(x))^{m_1}} + \frac{h_2(x)}{(1 - \Theta_2(x))^{m_2}} + \dots + \frac{h_l(x)}{(1 - \Theta_l(x))^{m_l}}$$

with  $\deg(h_i) < m_i \forall i \in [l]$ .

### 4.2 Solving Recurrences

**Theorem 4.2** Let  $p(x)$  and  $q(x)$  be polynomials with  $\deg(p(x)) < \deg(q(x))$  and  $q(x) = (1 - \theta_1 x)^{m_1} \dots (1 - \theta_k x)^{m_k}$  where  $m_1, m_2, \dots, m_k \in \mathbb{N}$  and  $\theta_1, \theta_2, \dots, \theta_k \in \mathbb{C}$  are distinct.

Then there exists polynomials  $A_1(x), \dots, A_k(x)$  with  $\deg(A_1) < m_1, \dots, \deg(A_k) < m_k$  such that  $[x^n] \frac{p(x)}{q(x)} = A_1(n)\theta_1^n + \dots + A_k(n)\theta_k^n$  for all  $n \geq 0$ .

Given a recurrence  $a_n = q_1 a_{n-1} + q_2 a_{n-2} + \dots + q_k a_{n-k}$ ,  $n \geq k$  and initial values for  $a_0, a_1, \dots, a_{k-1}$ , determine  $a_n$  explicitly.

The **characteristic polynomial** for such a recurrence is  $1 - q_1x - q_2x^2 - \dots - q_kx^k$ . Equivalently, it is  $1 + q_1x + q_2x^2 + \dots + q_kx^k$  for  $a_n + q_1a_{n-1} + q_2a_{n-2} \dots + q_ka_{n-k} = 0$ .

**Theorem 4.3** *Given such a recurrence, let  $A(x) = a_0 + a_1x + a_2x^2 + \dots$*

*Then  $A(x) = \frac{p(x)}{q(x)}$  where  $q$  is the characteristic polynomial and  $\deg(p) < k$ .*

**Proof 4.1** We need to show that  $A(x)q(x)$  is a polynomial with degree  $< k$ .

Let  $n \geq k$ . Then

$$\begin{aligned} [x^n]A(x)q(x) &= [x^n](a_0 + a_1x + a_2x^2 + \dots)(1 - q_1x - q_2x^2 - \dots - q_kx^k) \\ &= a_n - q_1a_{n-1} - q_2a_{n-2} - \dots - q_ka_{n-k} \\ &= 0 \end{aligned} \quad \text{(by definition of } a_n)$$

So then  $\deg(A(x)(q(x))) < k$  as required.  $\square$

Combining **Theorem 4.2** and **4.3**, we have

**Theorem 4.4**

$$\begin{aligned} a_n &= [x^n]A(x) \\ &= [x^n]\frac{p(x)}{q(x)} \\ &= A_1(n)\theta_1^n + \dots + A_k(n)\theta_k^n \end{aligned}$$

where  $\deg(p) < k$ ,  $q$  is the characteristic polynomial,  $\theta_1, \dots, \theta_j$  are distinct,  $m_1, \dots, m_j \in \mathbb{N}$ ,  $q(x) = (1 - \theta_1x)^{m_1} \dots (1 - \theta_jx)^{m_j}$  and  $A_i$  is a polynomial of degree  $< m_i$ .

**Example 4.2** Solve the recurrence defined by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -1 \\ a_2 &= 17 \\ a_n &= a_{n-1} + 8a_{n-2} - 12a_{n-3} \end{aligned}$$

The characteristic polynomial is

$$\begin{aligned} q(x) &= 1 - x - 8x^2 + 12x^3 \\ &= (1 - 2x)^2(1 + 3x) \end{aligned}$$

So  $\theta_1 = 2, \theta_2 = -3$  and  $m_1 = 2, m_2 = 1$ .

So we know that there are polynomials  $A_1(x), A_2(x)$  where  $\deg(A_1) < 2, \deg(A_2) < 1$  and  $a_n = A_1(n)2^n + A_2(n)(-3)^n$  for all  $n$ .

Let  $A_1(x) = \alpha x + \beta$  and  $A_2(x) = \gamma$ . Then  $a_n = (\alpha n + \beta)2^n + \gamma(-3)^n$ .

Using our values for  $a_0, a_1, a_2$ , we have

$$\begin{aligned} a_0 &= 1 &= \beta + \gamma \\ a_1 &= -1 &= 2(\alpha + \beta) - 3\gamma \\ a_2 &= 17 &= 4(2\alpha + \beta) + 9\gamma \end{aligned}$$

$\alpha = 1, \beta = 0, \gamma = 1$  is the only solution. So  $a_n = n2^n + (-3)^n$ .

### 4.3 Binary Trees

A binary tree is either empty or a root vertex together with a left child and a right child, each of which is a (possibly empty) binary tree. This can be represented as  $(\bullet, S_1, S_2)$ .

Let  $T$  be the set of binary trees and  $w(S) =$  the number of vertices in  $S$  for each  $S \in T$ . We can recursively define this as  $w(\epsilon) = 0$  and  $w(\bullet, S_1, S_2) = 1 + w(S_1) + w(S_2)$ .

Let  $T(x) = \Phi_T(x)$ . Thus  $[x^n]T(x)$  is the number of binary trees of  $n$  vertices.

We have  $T = \{\epsilon\} \cup \{\bullet\} \times T \times T$  unambiguously. Then

$$\begin{aligned} \Phi_T(x) &= \Phi_{\{\epsilon\}}(x) + \Phi_{\{\bullet\}}(x)\Phi_T(x)^2 \\ T(x) &= 1 + xT(x)^2 \\ xT(x)^2 - T(x) + 1 &= 0 \\ 4x^2T(x)^2 - 4xT(x) + 4x &= 0 \\ (2xT(x) - 1)^2 - 1 + 4x &= 0 \\ (1 - 2xT(x))^2 &= 1 - 4x \\ 1 - 2xT(x) &= \pm \left( 1 - 2 \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^{n+1} \right) \quad (\text{by assignment 3}) \end{aligned}$$

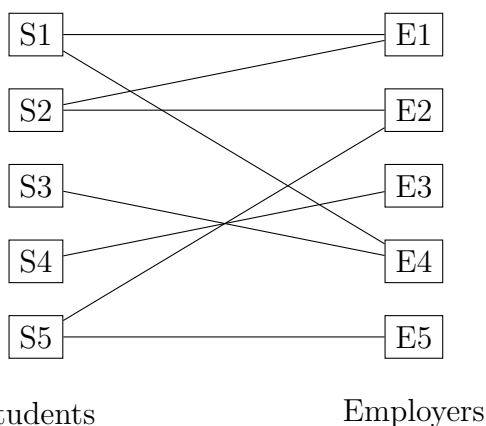
We cannot have the negative version since the LHS and the RHS would have different constant terms.

$$\begin{aligned} 1 - 2xT(x) &= 1 - 2 \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^{n+1} \\ T(x) &= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n \end{aligned}$$

Therefore there are  $\frac{1}{n+1} \binom{2n}{n}$  binary trees on  $n$  vertices.

## 5 Graph Theory

- Given a circuit diagram, can we make a flat circuitboard without edges crossing? (**Planarity**)
- How many colours are needed to colour each point in the plane so that no two points at distance 1 get the same colour?
- How many ways are there to drive between two intersections in Manhattan's one way system?
- Given some SE students and coop positions, where each position is compatible with only some students, can we give everyone a job?



- What is the cheapest way to get between two given cities?

### 5.1 Definitions

A **graph** is a pair  $(V, E)$  where  $V$  is a finite set and  $E$  is a set of unordered pairs of distinct elements of  $V$  (ie. two-element subsets of  $V$ ).

We call the elements of  $V$  the **vertices** and the elements of  $E$  the **edges**.

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ . An **isomorphism** from  $G_1$  to  $G_2$  is a bijection  $\phi : V_1 \rightarrow V_2$  such that for all  $u, v \in V_1$ ,  $\{u, v\} \in E_1$  if and only if  $\{\phi(u), \phi(v)\} \in E_2$ .

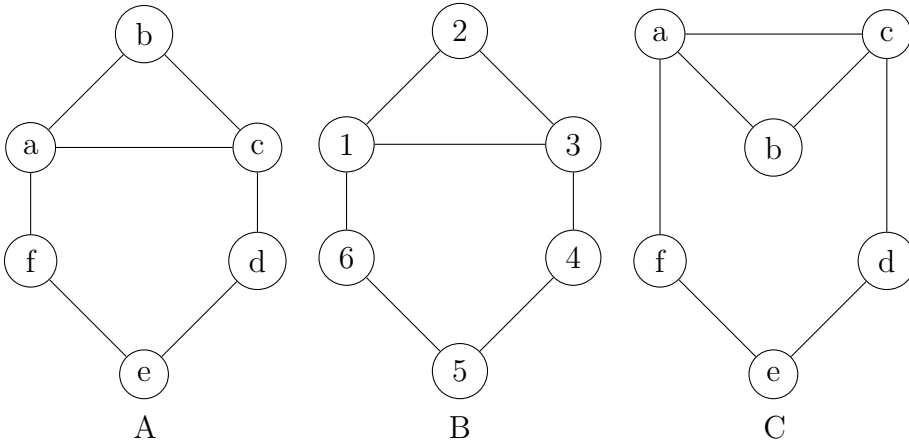
If an isomorphism exists then  $G_1$  and  $G_2$  are **isomorphic**. Graphs are isomorphic if they can be drawn in the same way.

We abbreviate an edge  $\{u, v\}$  by  $uv$ . If  $uv \in E$  then  $u$  and  $v$  are **adjacent** or **neighbours**.

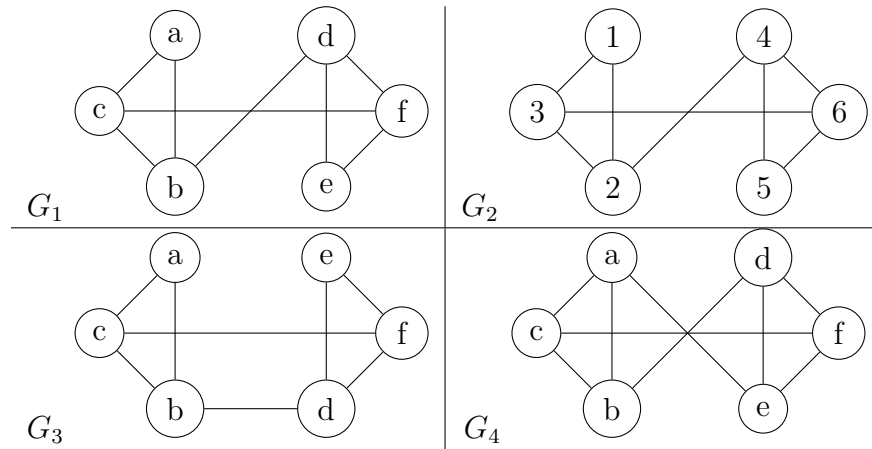
The **degree** of a vertex is its number of neighbours.

An edge  $uv$  is **incident** with vertices  $u$  and  $v$ .

**Example 5.1** Graphs  $A$  and  $B$  are the equal, although drawn differently.  $A$  and  $B$  are isomorphic but are not the equal since the vertices are labelled differently.



**Example 5.2**  $G_1$  and  $G_3$  are equal.  $G_2$  is not equal since the vertex names are different but isomorphic to  $G_1$  and  $G_3$ .  $G_4$  is not equal since it has an extra edge.



**Theorem 5.1** *Handshake Theorem*

$$\sum_{v \in V} \deg(v) = 2|E|$$

**Proof 5.1** Let  $S = \{(v, e) : v \text{ is incident with } e\}$ .

$$\begin{aligned} |S| &= \sum_{v \in V} (\# \text{ edges incident with } v) \\ &= \sum_{v \in V} \deg(v) \end{aligned}$$

Also

$$\begin{aligned} |S| &= \sum_{e \in E} (\# \text{ vertices incident with } e) \\ &= 2|E| \end{aligned}$$

So  $\sum_{v \in V} \deg(v) = 2|E|$ . □

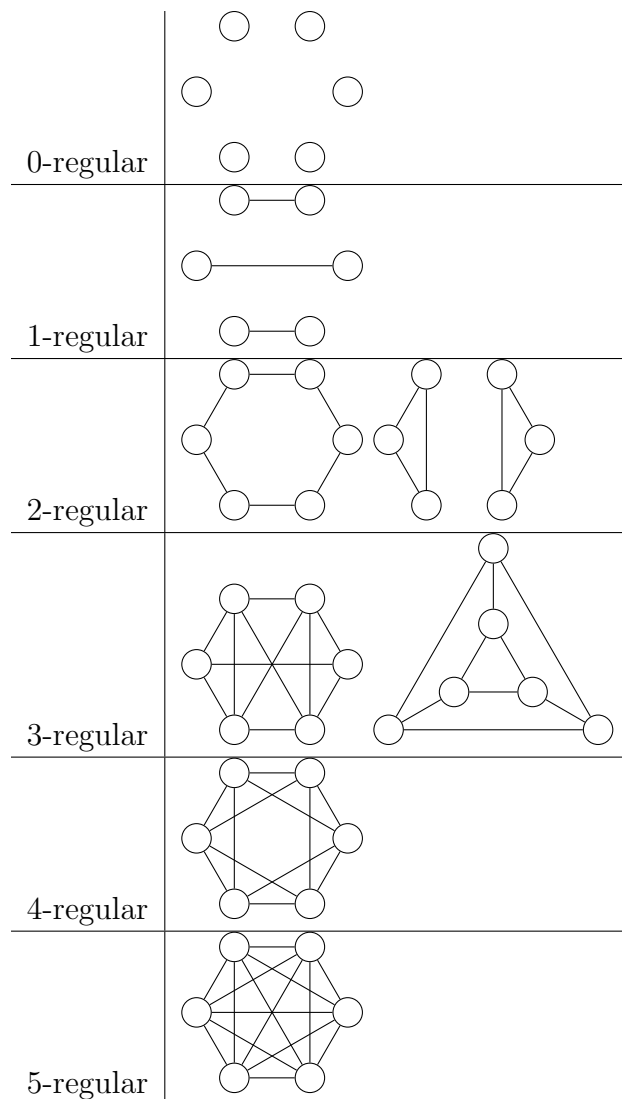
**Theorem 5.2** *Every graph has an even number of vertices of odd degrees.*

*This follows from the previous theorem. Since  $\sum_{v \in V} \deg(v) = 2|E|$  is even,  $\deg(v)$  is odd for an even number of  $v \in V$ .*

## 5.2 Regular Graphs

A graph is **regular** if every vertex has the same degree. If this degree is  $d$ , then the graph is called  $d$ -regular.

**Example 5.3** The following table shows all  $d$ -regular 6 vertex graphs.

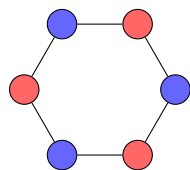


### 5.3 Bipartite Graph

A **bipartite graph** is a graph  $G = (V, E)$  for which there exists sets  $A, B$  such that  $A \cup B = V, A \cap B = \emptyset$  and every edge is incident with a vertex in  $A$  and a vertex in  $B$ .

$(A, B)$  is a **bipartition** of  $G$ .

**Example 5.4** A graph is bipartite if there is a 2-coloring for the graph. The following graph is bipartite since it has a 2-coloring.



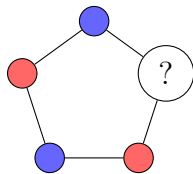
## 5.4 Cycle

A  **$k$ -cycle** is a graph  $C_k = (V, E)$  so that  $V$  has an ordering  $v_1, v_2, \dots, v_k$  so that  $E = \{v_1v_2, v_2v_3, \dots, v_{k-1}v_k, v_kv_1\}$ . So a  $k$ -cycle has  $k$  vertices and  $k$  edges.

**Theorem 5.3** *A  $k$ -cycle is bipartite if and only if  $k$  is even.*

**Proof 5.2** If  $k$  is even, then  $(\{v_1, v_3, v_5, \dots, v_{k-1}\}, \{v_2, v_4, \dots, v_k\})$  is a bipartition so  $C_k$  is bipartite.

If  $k$  is odd, WLOG suppose  $(A, B)$  is a bipartition with  $v_1 \in A$ . We show inductively that  $v_i \in A$  whenever  $i$  is odd. This is true for  $i = 1$ . If it is true for some  $v_i$  then since  $v_iv_{i+1} \in E$  and  $v_{i+1}v_{i+2} \in E$ , we have  $v_{i+1} \in B$  and  $v_{i+2} \in A$ . By induction,  $v_i \in A$  for all odd  $i$ . Thus  $v_k \in A$  and  $v_1 \in A$ . So since  $v_kv_1 \in E$ ,  $(A, B)$  is not a bipartition.

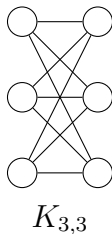


## 5.5 Complete Graph

A **complete graph**  $K_n$  is a graph  $G = (V, E)$  so that  $|V| = n$  and every pair of vertices is adjacent. A complete graph has  $\binom{n}{2}$  edges.

Only  $K_1$  and  $K_2$  are bipartite.

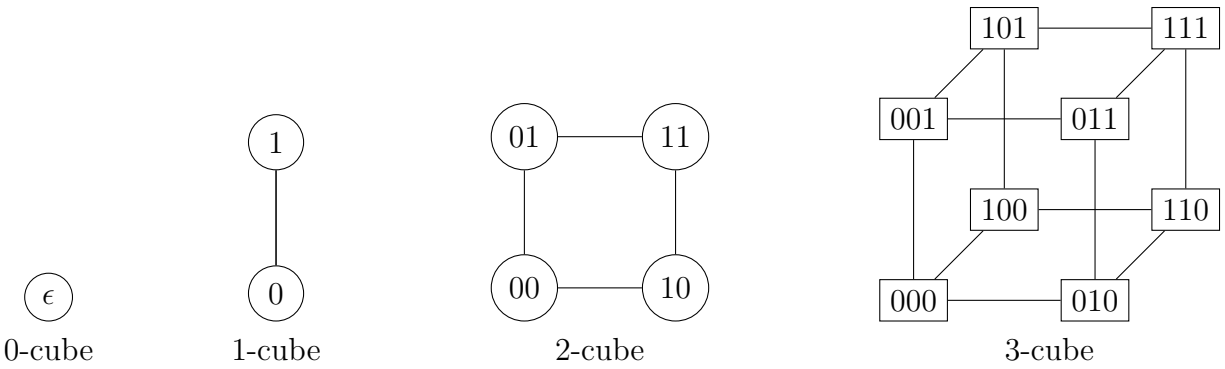
A **complete bipartite graph**  $K_{m,n}$  is a bipartite graph with bipartition  $(A, B)$  so that every vertex in  $A$  is adjacent to every vertex in  $B$  and  $|A| = m$  and  $|B| = n$ . From this, we get  $K_{m,n}$  has  $mn$  edges.



## 5.6 Cube

For  $n \geq 0$ , an  **$n$ -cube** is a graph with  $V = \{\text{binary strings of length } n\}$  in which two vertices are adjacent if they differ in exactly one position.





**Proof 5.3** The  $n$ -cube has  $2^n$  vertices and  $n2^{n-1}$  edges.

There are  $2^n$  vertices because there are  $2^n$  binary strings.

For each string  $s$  of length  $n$ , there are exactly  $n$  strings that differ from  $s$  in exactly one position. So each vertex of the  $n$ -cube has degree  $n$ . By the Handshake Theorem,

$$\begin{aligned} 2|E| &= \sum_{v \in V} \deg(v) \\ 2|E| &= |V|n \\ 2|E| &= n2^n \\ |E| &= n2^{n-1} \end{aligned}$$

In general, for a  $d$ -regular graph  $G$ , we have

$$\begin{aligned} 2|E| &= \sum_{v \in V} \deg(v) \\ 2|E| &= d|V| \\ |E| &= \frac{d|V|}{2} \end{aligned}$$

The  $n$ -cube can be constructed recursively from the  $(n - 1)$ -cube by taking two copies of the  $(n - 1)$ -cube and joining pairs of corresponding vertices with an edge.

**Proof 5.4** The  $n$ -cube is bipartite for all  $n$ .

Given a string with an even number of 1s, every neighbour will have an odd number of 1s. Therefore ( $\{\text{strings of length } n \text{ with an even number of 1s}\}, \{\text{strings of length } n \text{ with an odd number of 1s}\}$ ) is a bipartition of the  $n$ -cube for any  $n$ .

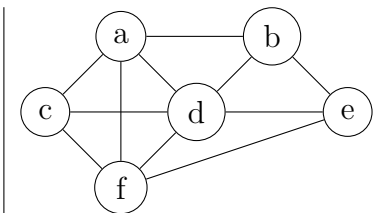
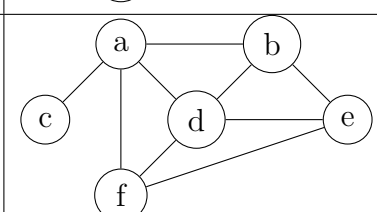
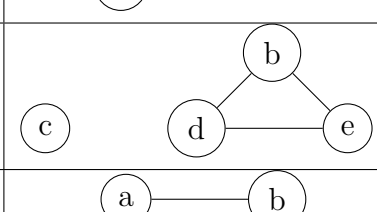
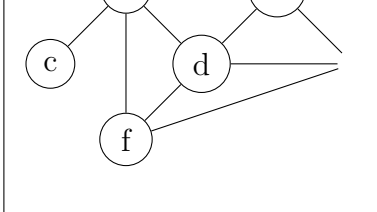
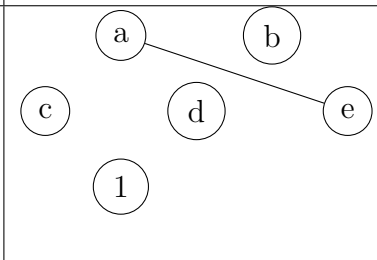
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material  
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## 5.7 Subgraph

A **subgraph** of a graph  $G = (V, E)$  is a graph  $G' = (V', E')$  where  $V' \subseteq V$  and  $E' \subseteq E$ . Essentially, it is a graph obtained by removed any number of edges or vertices from  $G$ .

A subgraph  $G' = (V', E')$  of  $G = (V, E)$  is a **spanning subgraph** of  $G$  if  $V' = V$ .

### Example 5.5

$G$		Original Graph
$H_1$		Subgraph of $G$
$H_2$		Subgraph of $G$
$H_3$		Not a subgraph of $G$ . It is not a graph due to edges to a nonexisting vertex.
$H_4$		Not a subgraph of $G$ . Contains $1 \in V'$ but $1 \notin V$ and $(a, e) \in E'$ but $(a, e) \notin E$ .

## 5.8 Walk

A **walk** of a graph  $G$  is an alternating series of vertices and edges  $v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k$  so that  $v_0, v_1, \dots, v_k \in V$  and each  $e_i$  is an edge of  $G$  from  $v_{i-1}$  to  $v_i$ . The **length** of this walk

is  $k$ , or the number of edges.

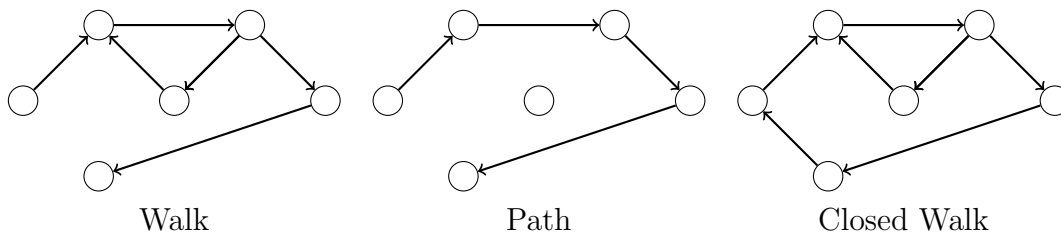
If  $v_0, v_1, \dots, v_k$  are distinct, the walk is also a **path**.

If  $v_0, v_1, \dots, v_k$  is a walk and  $v_0 = v_k$ , then the walk is **closed**.

If  $v_0, v_1, \dots, v_k$  is a closed walk and  $v_0, v_1, \dots, v_{k-1}$  are distinct, then the walk is a **cycle**.

A cycle that contains every vertex of a graph  $G$  is a **Hamilton cycle**. A graph with a Hamilton Cycle is **Hamiltonian**.

To specify a walk (or path) we often just list its vertices.



**Proof 5.5** If there is a walk from  $x$  to  $y$  in  $G$ , then there is also a path.

Let  $x = v_0, v_1, \dots, v_k = y$  be a **shortest** walk from  $x$  to  $y$  in  $G$ .

We argue that this walk is actually a path. Suppose it is not a path. Then there exists  $i, j$  such that  $0 \leq i < j \leq k$  and  $v_i = v_j$ .

But then  $v_0, v_1, \dots, v_i, v_{j+1}, v_{j+2}, \dots, v_k$  is a walk from  $x$  to  $y$  of length  $k - j + i < k$ . This contradicts the fact that the walk was as short as possible.

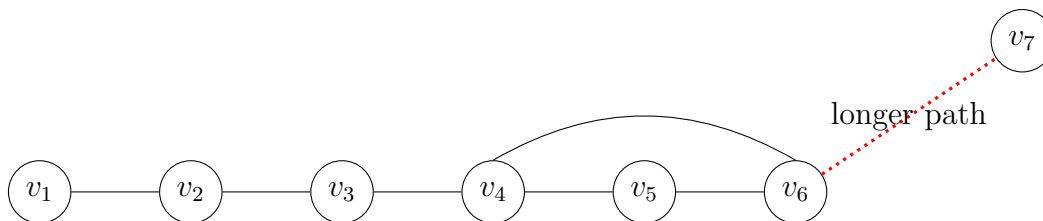
**Proof 5.6** If there is a path from  $x$  to  $y$  and a path from  $y$  to  $z$  in a graph  $G$ , then there is a path from  $x$  to  $z$  in  $G$ .

Let  $x = v_0, v_1, \dots, v_k = y$  and  $y = w_0, w_1, \dots, w_l = z$  be paths from  $x$  to  $y$  and  $y$  to  $z$  respectively. Now  $x = v_0, v_1, \dots, v_k = y = w_0, w_1, \dots, w_l = z$  is a walk from  $x$  to  $z$ . By what we proved above, there is a path from  $x$  to  $z$ .

**Proof 5.7** If  $G$  is a graph and every vertex has degree at least 2, then  $G$  has a cycle.

Let  $v_0, v_1, \dots, v_k$  be a longest path in  $G$ . Since the path is longest, every number of  $v_k$  is in  $\{v_0, v_1, \dots, v_{k-1}\}$ . Since  $\deg(v_k) \geq 2$ , there must be some  $0 \leq i \leq k - 2$  so that  $v_i$  is adjacent to  $v_k$  (if not then the path described is not the longest).

Now  $v_i, v_{i+1}, \dots, v_k, v_i$  is a cycle.



**Theorem 5.4** (Dirac 1952) *If a graph  $G$  has  $n \geq 3$  vertices and every vertex of  $G$  has degree  $\geq \frac{n}{2}$ , then  $G$  has a Hamilton cycle.*

**Proof 5.8** Let  $v_0, v_1, \dots, v_k$  be a longest path of  $G$ .

**Claim 1:** There is a cycle of  $G$  whose vertices are  $v_0, v_1, \dots, v_k$  (in some order).

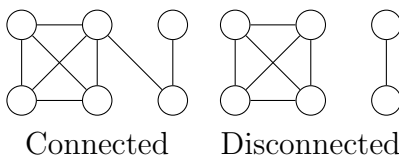
By maximality of the path, every neighbour of  $v_0$  and every neighbour of  $v_k$  lies in the path. Since  $v_0$  and  $v_k$  each have degree  $\geq \frac{n}{2}$ , we can find a neighbour  $v_l$  of  $v_k$  so that  $v_{l+1}$  is a neighbour of  $v_0$ . Then  $v_0, v_1, \dots, v_l, v_k, v_{k-1}, \dots, v_{l+1}, v_0$  is a cycle.

**Claim 2:** Every vertex of  $G$  is in  $\{v_0, \dots, v_k\}$ .

Since  $\{v_0, v_1, \dots, v_k\}$  contains  $v_0$  and all its neighbours,  $|\{v_0, \dots, v_k\}| \geq \frac{n}{2} + 1$ . If there is some  $w \in V$  such that  $w \notin \{v_0, \dots, v_k\}$  then since  $\deg(w) > \frac{n}{2}$ ,  $w$  has some neighbour in  $\{v_0, \dots, v_k\}$ . But now  $\{v_0, \dots, v_k, w\}$  contains a path of length  $k + 1$ , contradicting the maximality of the original path.

## 5.9 Connected

A graph  $G$  is **connected** if for all vertices  $x$  and  $y$ ,  $G$  contains a walk (or path) from  $x$  to  $y$ .



**Proof 5.9** If  $x$  is a vertex of a graph  $G$ , and for all vertices  $y$  of  $G$ , there is a path from  $x$  to  $y$ , then  $G$  is connected. (Note that this is a weaker statement than our definition).

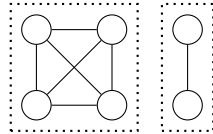
Let  $u, v$  be vertices of  $G$ . There is a walk from  $u$  to  $x$  and a walk from  $x$  to  $v$ , so there is a walk from  $u$  to  $v$ . Therefore,  $G$  is connected.

Which graphs are connected?

- Complete graphs
- Complete bipartite graphs are connected unless one side has no vertices (eg.  $K_{0,3}$ )

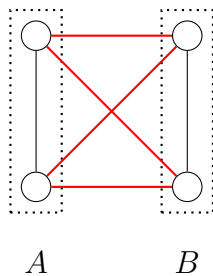
- Cycles
- Cubes

A **component** of a graph  $G$  is a maximal connected subgraph of  $G$ . That is, a connected subgraph  $H$  of  $G$  such that no connected subgraph  $H'$  of  $G$  has  $H$  as a proper subgraph.



## 5.10 Cut

Let  $(A, B)$  be a partition of the vertex set of a graph  $G$  ( $A \cup B = V$  and  $A \cap B = \emptyset$ ). The **cut** induced by  $(A, B)$  denotes the set of edges with one end in  $A$  and the other in  $B$ .



If the cut induced by  $(A, B)$  is the entire edge set, then  $(A, B)$  is a bipartition so the graph is bipartite. If  $A, B \neq \emptyset$  but the cut induced by  $(A, B)$  is empty, then graph is disconnected.

**Theorem 5.5** *Let  $G$  be a graph.  $G$  is connected if and only if there does not exist a partition  $(A, B)$  of  $V$  such that  $A, B \neq \emptyset$  and the cut induced by  $(A, B)$  is empty*

**Proof 5.10** Suppose that  $G$  is connected, but there exists a partition  $(A, B)$  of  $V$  inducing an empty cut with  $A \neq \emptyset$  and  $B \neq \emptyset$ .

Let  $u \in A, v \in B$ . By connectedness,  $G$  contains a path  $u = u_0, u_1, \dots, u_k = v$ . Note that  $u_0 \in A, v_k \in B$ . Let  $0 \leq i < k$  be maximal such that  $u_i \in A$ . By maximality,  $u_{i+1} \in B$ , so  $G$  contains an edge from  $A$  to  $B$ . This is a contradiction.

Conversely, suppose that  $G$  is disconnected. Let  $C$  be a component of  $G$ . Let  $V_C$  be the set of vertices in  $C$ . Since  $C$  is connected and  $G$  is not, we know that  $V_C \subsetneq V$  and  $V_C \neq \emptyset$  so  $(V_C, V \setminus V_C)$  is a partition of  $V$  into nonempty parts. Since  $C$  is a maximal connected subgraph, there is no edge from a vertex in  $V_C$  to one in  $V \setminus V_C$ .

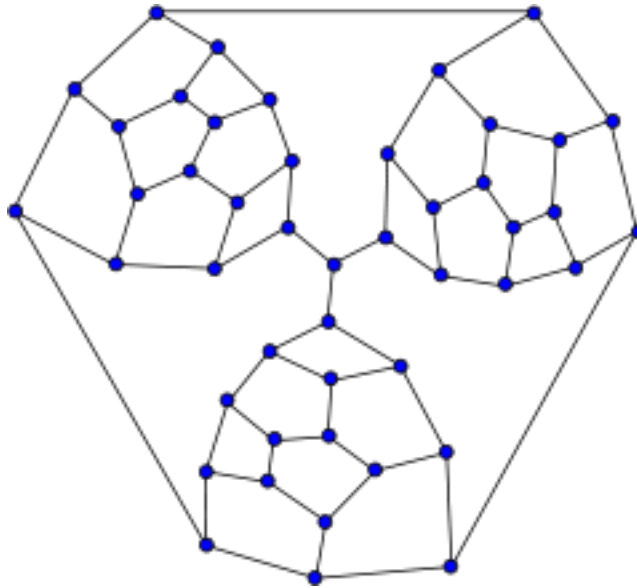
**Theorem 5.6** (Chvatal 1972)

If  $G$  is a graph whose vertices have degrees  $d_1 \leq d_2 \leq d_s \leq \dots \leq d_n$  and for each  $i \leq \frac{n}{2}$ , either  $d_i > i$  or  $d_{n-i} \geq n - i$ , then  $G$  is Hamiltonian.

For  $k \in \mathbb{N}$ , a graph is  **$k$ -connected** if for every pair of vertices  $u, v$  there are  $k$  internally disjoint paths from  $u$  to  $v$ .

**Tait Conjecture:** Every 3-connected graph that is planar is Hamiltonian.

The Tutte graph is a planar 3-connected graph but is not Hamiltonian.

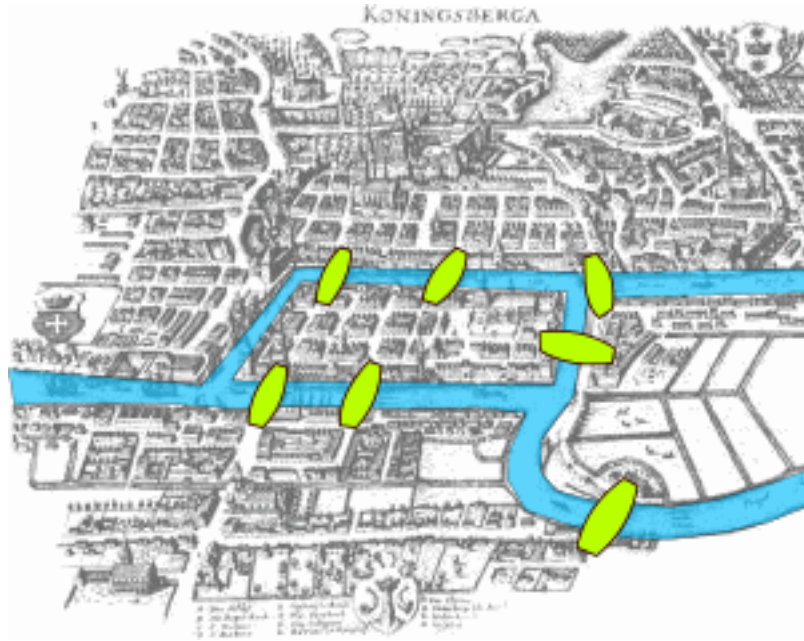


**Theorem 5.7** (Tutte)

Every 4-connected planar graph is Hamiltonian.

## 5.11 Euler Tour

Inspired by the problem, “Can we walk around Königsberg, crossing each bridge once, and returning to the start?”



An **Euler tour** in a graph is a closed walk containing each edge exactly once. A graph containing an Euler tour is Eulerian.

**Theorem 5.8** *If  $G$  has an Euler tour, then every vertex of  $G$  has even degree.*

**Proof 5.11** Let  $v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k = v_0$  be an Euler tour.

Let  $v$  be a vertex of  $G$ . Each occurrence of  $v$  in the sequence  $v_0, v_1, \dots, v_{k-1}$  has an edge both before and after it in the tour (where we consider  $e_k$  to be before  $v_0$ ). Since the tour includes each edge exactly once, this means that every such  $v$  has even degree.  $\square$

**Theorem 5.9** *If  $G$  is a connected graph in which every vertex has even degree, then  $G$  has an Euler Tour.*

**Proof 5.12** The theorem is trivial if there are no edges. Let  $m > 0$  and suppose inductively that the result holds for all graphs on  $< m$  edges.

Let  $G$  be a connected graph with  $m$  edges in which every vertex has an even degree. Let  $v_0, e_1, v_1, v_2, \dots, v_{k-1}, e_k, v_k = v_0$  be a closed walk of  $G$  with as many edges as possible. Let  $F = \{e_1, e_2, \dots, e_k\}$ .

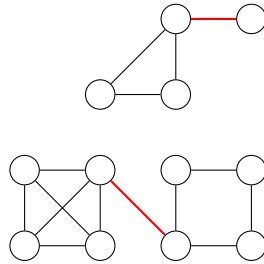
Since every vertex has even degree and  $G$  is connected, every vertex has degree  $\geq 2$  so  $G$  has a cycle (5.7). Therefore  $F$  contains at least as many edges as the cycle so  $F \neq \emptyset$ . If  $F = E$ , then the graph has an Euler Tour.

Then consider when  $F \neq E$ . Let  $H = (V, E \setminus F)$  be the subgraph of  $G$  formed by removing all edges in  $F$ . Since the subgraph  $(V, F)$  is Eulerian, every vertex is incident with an even

number of edges in  $F$ , so removing  $F$  gives a graph in which every vertex has even degree. Note since  $F \neq E$ , that  $H$  has  $> 1$  edge. Let  $C$  be a component of  $H$  that contains an edge. Now  $C$  is connected, has  $< m$  edges and every vertex has an even degree. So by the inductive hypothesis,  $C$  has an Euler Tour  $w_0, f_1, w_1, f_2, \dots, f_l, w_l = w_0$ . Since  $G$  is connected, there is a vertex  $x$  of  $C$  incident with an edge in  $F$ . Now we can adjoin the walks  $v_0, e_1, v_1, \dots, e_k, v_k$  and  $w_0, f_1, w_1, \dots, w_l$  at their common vertex  $x$  to create a closed walk not repeating edges in  $G$ . Such a walk is longer than our original one which is a contradiction.  $\square$

## 5.12 Bridges

An edge  $e$  is a **bridge** of a graph  $G$  if the graph  $G - e$  has more components than  $G$ . If  $G = (V, E)$  then  $G - e$  is the graph  $(V, E \setminus \{e\})$ .



**Theorem 5.10**  $e = uv$  is a bridge of a graph  $G$  iff  $u$  and  $v$  are in different components of  $G - e$ .

**Theorem 5.11** An edge  $e = uv$  is a bridge of a graph  $G$  iff it is not contained in a cycle of  $G$ .

**Proof 5.13** Suppose  $e$  is contained in a cycle  $C$ . Then the edges in  $C - e$  form a path from  $u$  to  $v$  in  $G - e$  so  $u$  and  $v$  are in the same component of  $G - e$ . Then by 5.10,  $e$  is not a bridge.

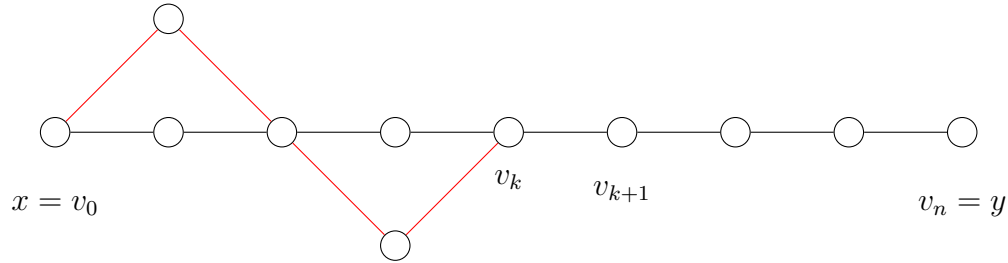
All of these implications work in reverse so we can prove the converse in a similar manner.  $\square$

**Proof 5.14** If  $x$  and  $y$  are vertices of a connected graph  $G$  with no bridge, then  $G$  contains two edge-disjoint paths from  $x$  to  $y$ .

Let  $x = v_0, v_1, \dots, v_n = y$  be a path from  $x$  to  $y$  in  $G$ . Let  $k \in \{0, 1, \dots, n\}$  be maximal so that  $G$  contains two edge disjoint paths from  $x$  to  $v_k$ . If  $v_k = v_n$ , the theorem holds so suppose  $k < n$ . Let  $P, P'$  be edge-disjoint paths from  $x$  to  $v_k$ . The edge  $v_k v_{k+1}$  is not a bridge so there is some path  $Q'$  from  $v_{k+1}$  to  $v_k$  that does not contain the edge  $v_k v_{k+1}$ .



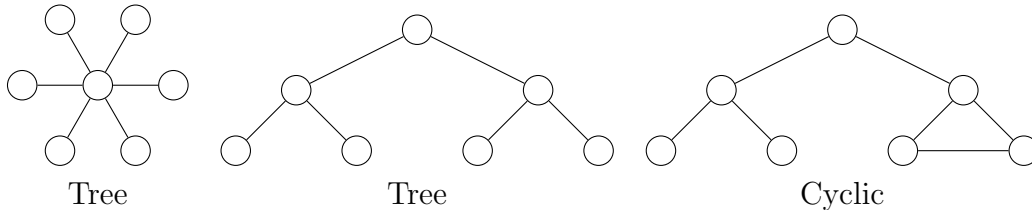
Let  $w$  be the first vertex of  $Q'$  that is contained in  $P \cup P'$  and let  $Q$  be the subpath of  $Q'$  from  $v_{k+1}$  to  $w$ . Now the edges in  $P, P', Q$  and  $\{v_k v_{k+1}\}$  contain edge-disjoint paths from  $x$  to  $v_{k+1}$  contradicting the maximality of  $k$ .



### 5.13 Trees

A **tree** is a connected graph with no cycles (acyclic graph).

A **leaf** of a tree is a degree-1 vertex.



**Proof 5.15** A connected graph  $G$  is a tree iff every edge is a bridge.

We saw in 5.11 that an edge is a bridge iff it is contained in no cycle. This result follows.  $\square$

**Proof 5.16** Every tree on  $\geq 2$  vertices has  $\geq 2$  leaves.

Let  $v_0, v_1, \dots, v_k$  be a longest path. By maximality, every neighbour of  $v_0$  or  $v_k$  is in the path. By acyclicity,  $v_0$  and  $v_k$  have only neighbours  $v_1, v_{k-1}$  respectively. So  $\deg(v_0) = \deg(v_k) = 1$ . Then  $v_0, v_k$  are leaves.  $\square$

**Proof 5.17** If  $T$  is a tree on  $n$  vertices, then  $T$  has  $n - 1$  edges.

Trivial if  $n = 1$ . Suppose that the statement holds for every tree on  $k$  vertices for some  $k > 1$ . Let  $T$  be a tree on  $k + 1$  vertices. Let  $v$  be a leaf of  $T$  and let  $T'$  be the graph obtained by removing  $v$  and a single incident edge from  $T$  (by 5.16).

$T'$  is acyclic since  $T$  is acyclic. If  $x, y$  are vertices of  $T'$ , then by connectedness of  $T$ , there is a path of  $T$  from  $x$  to  $y$ . Since  $\deg(v) = 1$  this path does not contain  $v$  so it is also a path of  $T'$ . Therefore  $T'$  is connected and is a tree.  $T'$  has  $k$  vertices so it has  $k - 1$  edges. Therefore  $T$  has  $k$  edges as required.  $\square$

**Proof 5.18** Trees are bipartite.

Prove by removing a leaf and using induction. □

A **spanning tree** of a connected graph  $G$  is a subgraph of  $G$  that is a tree with the same vertex set as  $G$ .

**Proof 5.19** Every connected graph has a spanning tree.

Let  $G = (V, E)$  be a connected graph. Let  $F$  be a minimal subset of  $E$  so that the graph  $H = (V, F)$  is connected.

Since  $F$  is minimal, the graph  $H - e$  is disconnected for every  $e \in F$ , so every edge of  $H$  is a bridge. Thus  $H$  is a tree so it is a spanning tree. □

**Proof 5.20** A graph  $G$  is bipartite iff it contains no odd cycle.

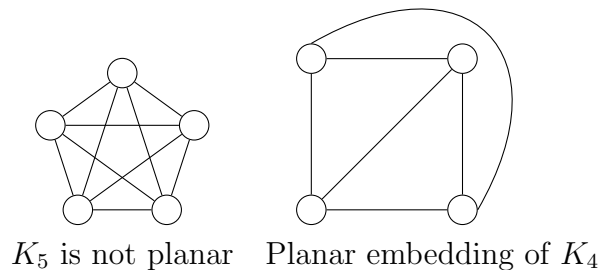
We may assume that  $G$  is connected. If not, then consider each component individually. Let  $T$  be a spanning tree of  $G$ . Suppose  $G$  has no odd cycles. We know trees are bipartite. Let  $(A, B)$  be a bipartition of  $T$ .

We'll show that  $(A, B)$  is also a bipartition of  $G$ . Suppose otherwise. Let  $x, y$  be adjacent vertices of  $G$  that are both in  $A$  or both in  $B$ . Let  $x = u_0, \dots, u_k = y$  be a path from  $x$  to  $y$  in  $T$ .

Since each edge of  $T$  has an end in  $A$  and an end in  $B$ , vertices in this path alternate between  $A$  and  $B$ . The ends are in the same set, so the length  $k$  is even.  $x = u_0, u_1, \dots, u_k = y = x$  is an odd cycle of  $G$ , a contradiction. We proved the converse earlier in Proof 5.2. □

## 5.14 Planar Graph

A **drawing** of a graph  $G$  is a subset of the plane such that every vertex corresponds to a distinct point, every edge corresponds to an open arc and the closure of each edge is exactly its endpoints.



We we draw any graph in the plane such that edges only meet at vertices?

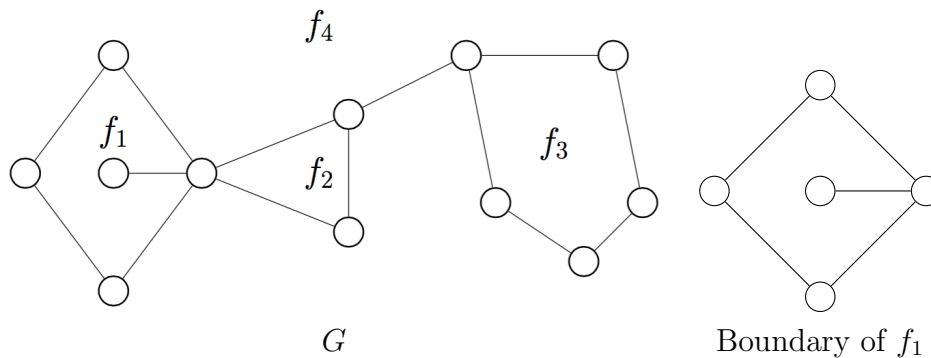
A graph  $G$  is **planar** if there is a drawing of  $G$  in the plane so that every vertex  $B$  is mapped to a distinct point and the intersections of the edges are disjoint. Such a drawing is called a **planar embedding** of  $G$  or a **planar map**.

Note if  $G$  is disconnected then  $G$  is planar iff every component of  $G$  is planar.

**Theorem 5.12** (*Fary's Theorem*) *If  $G$  is planar then  $G$  can be embedded in the plane using only straight lines.*

If  $G$  is embedded in the plane  $P$ , the closures of the connected components of  $P \setminus G$  are the **faces** of the embedding. The unbounded face of an embedding is called the **outer face**.

The subgraph of  $G$  formed by the vertices and edges in the bounding of  $F$  is the **boundary** of  $F$ .



A vertex or edge of  $G$  in the boundary of  $F$  is **incident** with  $F$ . As we “walk” along the boundary of  $F$  we set a closed walk in  $G$ . Such a walk is the **boundary walk** of  $F$  denoted  $W_F$ .

The **degree** of  $F$  is the length of  $W_F$  (number of edges in  $W_F$ ).

Any edge  $e$  appears twice in the set of boundary walks for faces of  $G$  since  $e$  is part of the boundary of two faces (could be the same face twice).

Given  $G$  embedded in the plane, the bridges of  $G$  are exactly the edges that appear twice in some face boundary walk.

**Proof 5.21** All trees  $T$  are planar.

In any embedding of  $T$  in the plane, we have exactly one face. And any edge of  $T$  is contained in the boundary walk twice. So  $\deg(F) = 2|E(T)| = 2|V(T)| - 2$ .

**Theorem 5.13** (*Handshake Theorem for Faces*)

If we have a planar embedding of a connected graph  $G$  with faces  $F_1, F_2, \dots, F_k$ , then

$$\sum_{i=1}^k \deg(F_i) = 2|E(G)|$$

**Theorem 5.14** (*Euler's Formula*)

Let  $G$  be a connected graph with  $v$  vertices and  $e$  edges. If  $G$  has an embedding in the plane with  $f$  faces, then

$$v - e + f = 2$$

**Proof 5.22** For a connected graph  $G$  with  $v$  vertices, the minimum number of edges in  $G$  is  $v - 1 = e$  when  $G$  is a tree. Any embedding of a tree in the plane has one face. Then

$$v - e + f = v - (v - 1) + 1 = 2$$

Suppose the claim is true for graphs on  $v$  vertices and  $< e$  edges (with  $e \geq v$ ). Since  $e \geq v$ ,  $G$  is not a tree and there is some edge of  $G$  that is not a bridge.

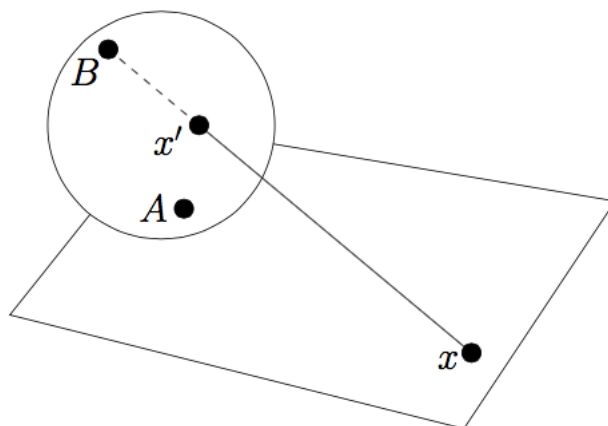
Suppose  $\{a, b\}$  is a non-bridge edge of  $G$ . Consider  $H = G \setminus \{a, b\}$ .  $H$  has  $v$  vertices,  $e - 1$  edges and  $H$  is connected. We showed earlier that an edge separates two faces and if the edge is not a bridge, then the two faces are different. Then by removing the edge, we join the two faces. So  $H$  has  $f - 1$  faces.

Then by the inductive hypothesis

$$\begin{aligned} v - e + f &= 2 \\ v - (e - 1) + (f - 1) &= 2 \end{aligned}$$

as required. □

### 5.14.1 Stereographic Projection



Any drawing on the plane can be converted to a drawing on a sphere via a stereographic projection. We'll have the sphere tangent to the plane at point  $A$  with point  $B$  antipodal to  $A$  on the sphere. Then any point  $x'$  on the sphere other than  $B$  can be mapped to a point  $x$  on the plane. If join  $B$  and  $x'$  with a line, we can have  $x$  be the intersection between the plane and the line.

Then a sphere minus a single point is equivalent to a plane. Our point  $B$  on the sphere cannot be mapped to a point on the plane and is a point on the plane at "infinite distance".

**Theorem 5.15** *A graph is planar if and only if it can be drawn on a sphere.*

### 5.14.2 Platonic Graphs

A **fullerene** is a planar 3-regular graph with an embedding containing only degree 5 or 6 faces.

**Proof 5.23** All fullerenes have exactly 12 degree 5 faces.

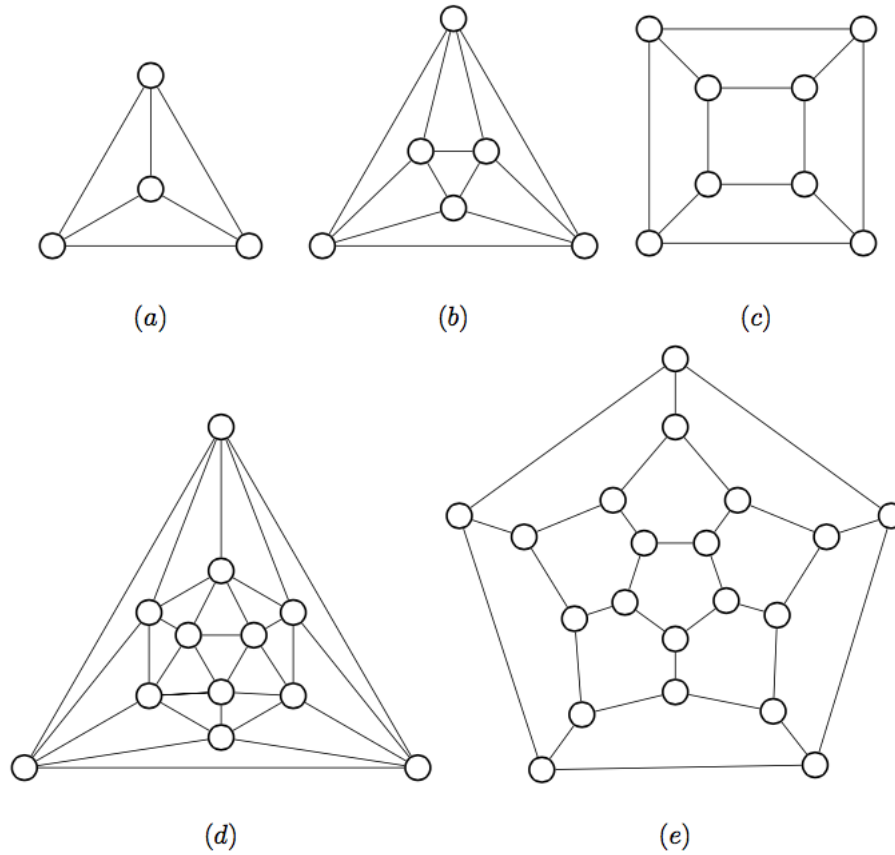
Let  $f_5$  be the number of degree 5 faces and  $f_6$  be the number of degree 6 faces. Then  $f = f_5 + f_6$  by the definition of a fullerene.

By Euler's formula,  $v - e + f_5 + f_6 = 2$ . Theorem 5.13 gives  $5f_5 + 6f_6 = 2e$ . Then since a fullerene is 3-regular and by the Handshake Theorem we have  $v - \frac{3}{2}v + f_5 + f_6 = 2$  and  $5f_5 + 6f_6 = 3v$ . Rearranging and equating  $f_6$  in each equation gives

$$\begin{aligned} \frac{3v - 5f_5}{6} &= 2 + \frac{1}{2}v - f_5 \\ f_5 &= 12 \end{aligned}$$

□

A graph is **platonic** if it is  $d$ -regular (with  $d \geq 3$ ) and has an embedding in the plane where all faces have degree  $d^*$  with  $d^* \geq 3$ .



The 5 platonic graphs.

**Theorem 5.16** *There are exactly 5 platonic graphs.*

**Proof 5.24** A platonic graph  $G$  has  $(d, d^*) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$ .

Since  $G$  is  $d$ -regular and all faces have degree  $d^*$ ,

$$\begin{aligned} dv &= 2e \\ v &= \frac{2e}{d} \\ d^*f &= 2e \\ f &= \frac{2e}{d^*} \end{aligned}$$

By Euler's formula,

$$\begin{aligned} \frac{2e}{d} - e + \frac{2e}{d^*} &= 2 \\ \frac{2}{d} + \frac{2}{d^*} &= \frac{2}{e} + 1 \end{aligned}$$

For any  $e$ ,  $1 + \frac{2}{e} > 1$ .

Then if  $d \geq 4$  and  $d^* \geq 4$ , then  $\frac{2}{d} + \frac{2}{d^*} \leq 1$ . If  $d = 3$  and  $d^* \geq 6$ , then  $\frac{2}{d} + \frac{2}{d^*} \leq 1$ . This is a contradiction.  $\square$

**Proof 5.25** We'll prove that there are 5 platonic graphs.

If  $G$  is platonic with vertex degree  $d$  and face degree  $d^*$ ,

$$e = \frac{2dd^*}{2d + 2d^* - dd^*}$$

This can be shown using Euler's formula and that  $v = \frac{2e}{d}$  and  $f = \frac{2e}{d^*}$ .

So for each  $(d, d^*)$  we have  $v, e, f$  as determined. Each tuple gives one platonic graph.

**Proof 5.26** If  $G$  is connected and not a tree, then the boundary of every face in a planar embedding of  $G$  contains a cycle.

Since  $G$  has a cycle, it has more than one face. Therefore, every face  $f$  is adjacent to at least one other face  $g$ .

Let  $e = v_0v_1$  be an edge that is incident with both  $f$  and  $g$ . Let  $H$  be the component in the boundary graph of face  $f$  containing the edge  $e_1$ . Let

$$W_f = (v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_{n-1}, v_0)$$

be the boundary walk of  $f$ . Since the edge  $e_1$  is incident with both  $f$  and  $g$ , it is contained in  $W_f$  precisely once.

The edge  $e_1$  is not a bridge of  $H$  because  $(v_1, e_2, v_2, \dots, v_{n-1}, e_{n-1}, v_0)$  is a walk from  $v_1$  to  $v_0$  in  $H - e_1$ . Therefore  $H$  contains a cycle.  $\square$

To prove a graph is non-planar, we usually prove a property true for all planar graphs and then show that a graph does not have this property.

**Proof 5.27** If  $G$  is a connected planar graph with  $p \geq 3$  vertices and  $q$  edges, then  $q \leq 3p - 6$ .

If  $G$  is a tree, then the statement holds because  $q = p - 1$ . If  $G$  is not a tree, consider a planar embedding of  $G$  with  $p$  vertices,  $q$  edges and  $r$  faces. By the Handshake theorem for faces,

$$2q = \sum_{f \in F} \deg(f)$$

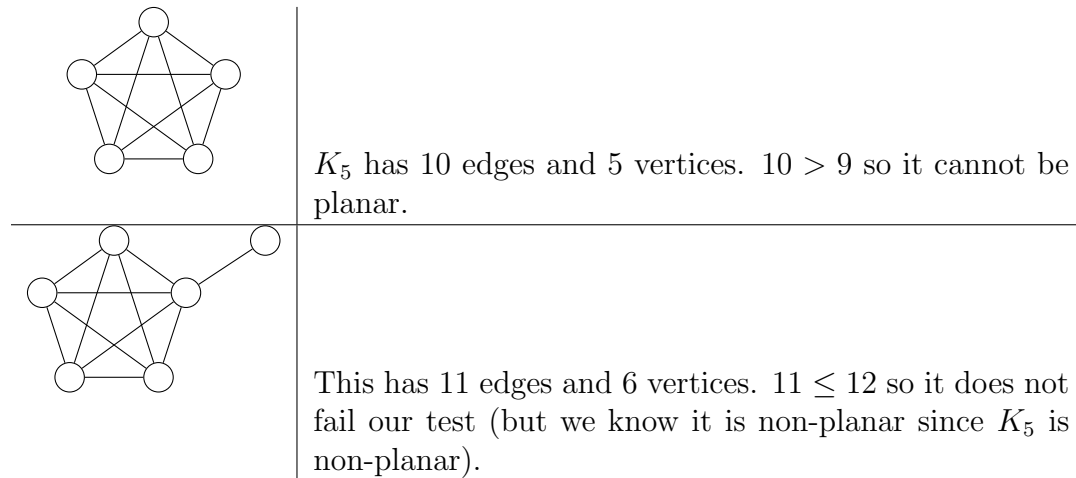
. Each face has degree  $\geq 3$  since the boundary of every face contains a cycle. Then

$$\begin{aligned} 2q &\geq 3r \\ r &\leq \frac{2}{3}q \end{aligned}$$

By Euler's Formula,

$$\begin{aligned}
 2 &= p - q + r \\
 2 &\leq p - q + \frac{2}{3}q \\
 2 &\leq p - \frac{1}{3}q \\
 q &\leq 3p - 6
 \end{aligned}$$

□



**Proof 5.28** If  $G$  is a connected planar graph that is not a tree with  $p$  vertices,  $q$  edges and every cycle has length  $\geq d$ , then  $q \leq \frac{d}{d-2}(p-2)$ .

Since every face boundary contains a cycle,  $\deg(f) \geq d$  for each face  $f$ . By handshaking,  $2q = \sum \deg(f) \geq dr$  so  $r \leq \frac{2}{d}q$ .

By Euler's formula

$$\begin{aligned}
 2 &= p - q + r \\
 2 &\leq p - q + \frac{2}{d}q \\
 q(1 - \frac{2}{d}) &\leq p - 2
 \end{aligned}$$

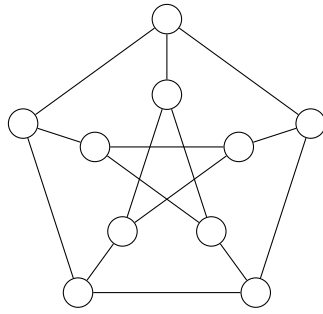
So  $q \leq \frac{d}{d-2}(p-2)$

□

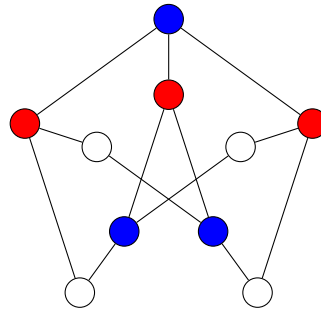
Then  $K_{3,3}$  is non-planar since every cycle has length at least 4, it has 9 edges and 6 vertices.

Is the Petersen graph planar? We can remove some edges from it. Notice that the graph below is homeomorphic to  $K_{3,3}$ , which is non-planar. Then the Petersen graph is non planar since it contains  $K_{3,3}$  which is non-planar.

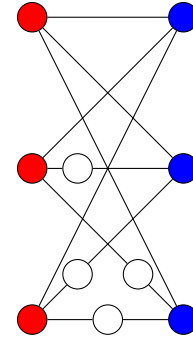




Petersen Graph



Petersen Graph with edges removed



Subdivision of  $K_{3,3}$

A **subdivision** of a graph  $G$  is a graph obtained by replacing each edge of  $G$  by a path of length  $\geq 1$ .

**Theorem 5.17** *If  $H$  is a subdivision of a graph  $G$ , then  $H$  is planar iff  $G$  is planar.*

As a corollary, if  $H$  is a nonplanar graph and  $G$  is a graph containing a subdivision of  $H$  as a subgraph, then  $G$  is nonplanar.

**Theorem 5.18** *(Kuratowski's Theorem)*

*$G$  is planar iff  $G$  contains no subdivision of  $K_5$  or  $K_{3,3}$  as a subgraph.*

## 5.15 Graph Coloring

Let  $k \in \mathbb{N}$ . A  **$k$ -coloring** of a graph  $G = (V, E)$  is a function from  $V$  to a set of size  $k$  (whose elements are called **colors**) so that adjacent vertices are mapped to different colors always.

A graph with a  $k$ -coloring is  **$k$ -colorable**.

$G$  is bipartite iff  $G$  is 2-colorable. The complete graph  $K_n$  is  $n$ -colorable but not  $(n - 1)$ -colorable. The cycle  $C_n$  is 2-colorable iff  $n$  is even and is 3-colorable if  $n$  is odd.

**Theorem 5.19** *(Four Colour Theorem)*

*Every planar graph is 4-colourable.*

The proof for the Four Colour Theorem is hard to prove. We'll prove the six-colour theorem instead by first proving the following lemma.

**Proof 5.29** Every planar graph has a vertex of degree  $\leq 5$ .

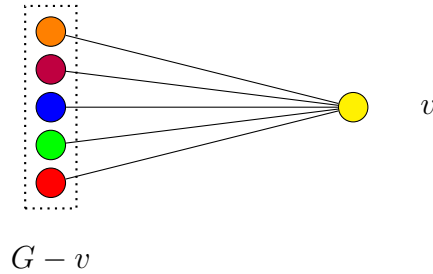
Let  $G = (V, E)$  be a planar graph. We know that  $|E| \leq 3|V| - 6$  by Proof 5.27. The handshake theorem shows that

$$\begin{aligned} \sum_{v \in V} \deg(v) &= 2|E| \\ \frac{\sum_{v \in V} \deg(v)}{|V|} &= \frac{2|E|}{|V|} \\ \frac{\sum_{v \in V} \deg(v)}{|V|} &\leq \frac{2(3|V| - 6)}{|V|} \frac{\sum_{v \in V} \deg(v)}{|V|} \leq 6 - \frac{12}{|V|} \end{aligned}$$

Then the average degree is  $\leq 6 - \frac{12}{|V|} < 6$  so  $G$  has a vertex of degree  $\leq 5$ . □

**Proof 5.30** Prove the six-colour theorem by induction on number of vertices. If  $G$  has  $\leq 6$  vertices, it is trivial. Suppose for  $n \geq 6$ , the theorem holds for every planar graph on  $n$  vertices. Let  $G'$  be a planar graph on  $n + 1$  vertices.

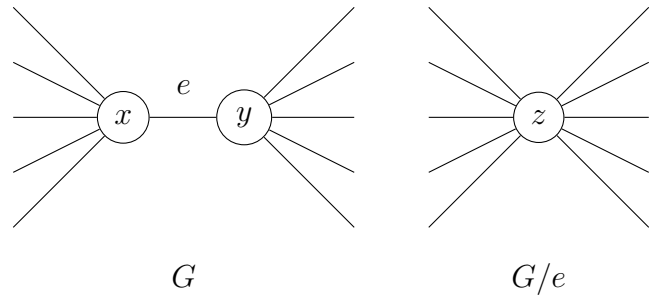
Let  $v$  be a vertex of degree  $\leq 5$  by Proof 5.29. Inductively,  $G - v$  has a 6-colouring. Some colour is not used by any neighbour of  $v$  since it has less than 5 adjacent vertices. Assigning this colour to  $v$  gives a 6-colouring of  $G$ . □



There exists a  $v$  in a planar graph with degree  $\leq 5$ .

### 5.15.1 Contraction

If  $e = xy$  is an edge of a graph  $G = (V, E)$  then  $G/e$  denotes the graph with vertex set  $(V \setminus \{x, y\}) \cup \{z\}$  where  $z$  is a new vertex not in  $V$  and edge set  $\{uv : uv \in E \text{ and } \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{wz : wx \in E \text{ or } wy \in E, w \notin \{x, y\}\}$ .



Note that the contraction of a planar graph is also planar.

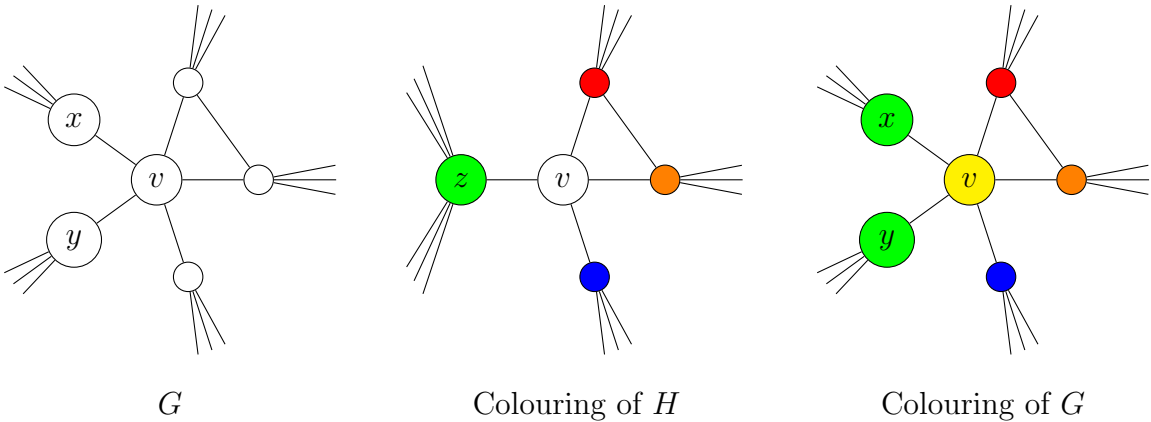
**Theorem 5.20** *Every planar graph is 5-colourable.*

**Proof 5.31** We'll prove by induction on number of vertices. It is trivial if  $|V| \leq 5$ . Suppose the result is true for every graph on  $\leq n$  vertices where  $n \geq 5$ . Let  $G$  be a graph on  $n + 1$  vertices. Let  $v$  be a vertex of  $G$  of degree  $\leq 5$ .

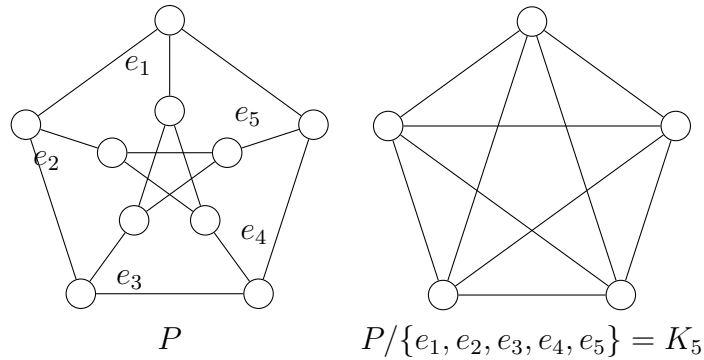
Consider the case when  $\deg(v) \leq 4$ . Inductively,  $G - v$  has a 5-colouring. Some colour is not used by any neighbour of  $v$  in this colouring. Assigning that colour to  $v$  gives a 5-colouring of  $G$ .

Now consider when  $\deg(v) = 5$ .  $G$  has no  $K_5$  subgraph since it is planar. Then there are neighbours  $x, y$  of  $v$  that are nonadjacent in  $G$ . Let  $e = xv, f = yv, H = G/e/f$ .  $H$  is planar and less vertices than  $G$  so it has a 5-colouring. Suppose that the vertex  $z$  to which  $x, y, v$  are identified is assigned colour  $c$  in this colouring of  $H$ .

Now assigning  $c$  to  $x$  and  $y$  and colouring every vertex in  $V \setminus \{x, y, v\}$  according to the colour it receives in  $H$  gives a colouring of  $G - v$  in which  $x$  and  $y$  both have the same colour. Now the neighbours of  $v$  use  $\leq 4$  colours in this colouring of  $G - v$  so we extending it to a 5-colouring of  $G$  as before. □



We can show the Petersen graph is nonplanar using contractions.



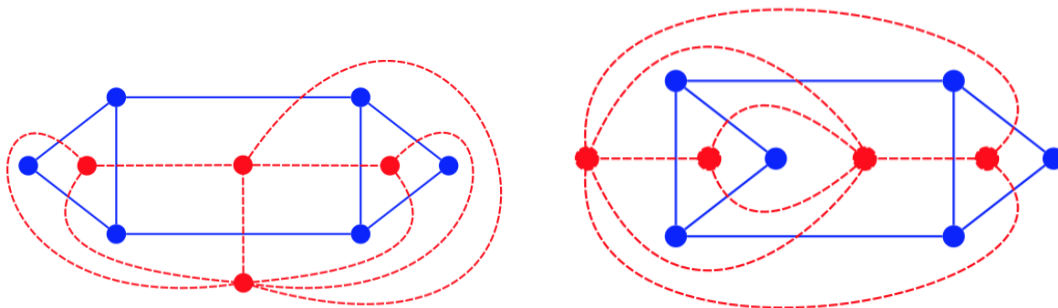
The **deletion** of an edge  $e$  in a graph  $G$  written  $G \setminus e$  is the graph obtained by removing  $e$  from  $G$ . It can also be written  $G - e$ .

**Theorem 5.21** *Kuratowski's Theorem (Minor Version)*

$G$  is planar iff neither  $K_5$  nor  $K_{3,3}$  can be obtained from  $G$  by contracting/deleting edges and removing vertices.

**5.15.2 Planar Dual**

Let  $G$  be a *connected planar embedding* of a graph. The **planar dual** of  $G$  is the graph  $G^*$  such that the set of vertices of  $G^*$  is the set of faces of  $G$  and two vertices of  $G^*$  are joined by an edge iff the corresponding faces are adjacent in  $G$ .



1.  $G^*$  has a drawing on top of  $G$  so that each edge of  $G^*$  crosses exactly one edge of  $G$  and each vertex of  $G^*$  is drawn inside its corresponding face.
2. Each edge of  $G^*$  corresponds naturally to a unique edge of  $G$ . In particular,  $G$  and  $G^*$  have the same number of edges.
3. The faces of  $G^*$  correspond naturally to vertices of  $G$ .
4.  $G^{**} = G$  if  $G$  (requires connectedness of  $G$ )

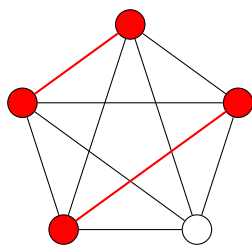
5.  $(G/e)^* = G^* \setminus e$  and  $(G \setminus e)^* = G^*/e$
6.  $G^*$  may have multiple edges or loops when  $G$  does not.
7. Different embeddings of  $G$  may have nonisomorphic duals (see graphs above)
8. Platonic graphs come in dual pairs.

## 5.16 Matchings and Covers

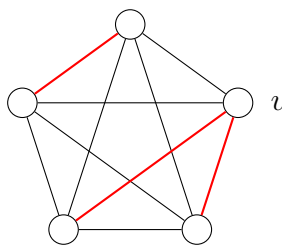
Given a graph  $G = (V, E)$ , a **matching** of  $G$  is a set  $M \subseteq E$  so that each vertex of  $G$  is incident with at most one edge in  $M$ .

A vertex incident with an edge of  $M$  is **saturated**. If a vertex is not incident with an edge it is **unsaturated**.

If every vertex is saturated, the  $M$  is a **perfect matching**.



Edges in the matching and saturated vertices are in red



Not a matching since there are two incident edges to  $v$

If  $M$  is a matching of a graph  $G$ , a path  $v_0, v_1, \dots, v_k$  of  $G$  is an  **$M$ -alternating path** if either  $v_i v_{i+1} \in M$  iff  $i$  is even or  $v_i v_{i+1} \in M$  iff  $i$  is odd.

If  $v_0 v_1 \notin M$  and  $v_{k-1} v_k \notin M$  and  $v_0, v_k$  are unsaturated then the  $M$ -alternating path is an **augmenting path**. Note that every augmenting path has odd length.

**Proof 5.32** If  $M$  is a matching of a graph  $G$  and  $M$  has an augmented path, then  $M$  is not a maximum matching of  $G$ .

If  $v_0, v_1, \dots, v_k$  is an augmenting path, then  $(M \setminus \{v_1 v_2, v_3 v_4, \dots, v_{k-2} v_{k-1}\}) \cup \{v_0 v_1, v_2 v_3, \dots, v_{k-1} v_k\}$  is a matching of  $G$  of size  $|M| + 1$ . So  $M$  is not a maximal matching.  $\square$

A **cover** of a graph  $G = (V, E)$  is a set  $C \subseteq V$  so that every edge of  $G$  is incident with a vertex in  $C$ .

Note that the vertex set  $V$  is trivially a cover. In a bipartite graph with bipartition  $(A, B)$ , both  $A$  and  $B$  are covers.

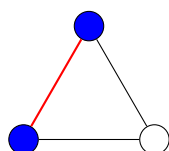
**Proof 5.33** If  $M$  is a matching of  $G$  and  $C$  is a cover of  $G$ , then  $|M| \leq |C|$ .

Since  $C$  is a cover, it contains at least one end from each edge in  $M$ . The ends of these edges are all distinct so  $|C| \geq |M|$ .  $\square$

**Proof 5.34** If  $M$  is a matching and  $C$  is a cover of  $G$ , and  $|M| = |C|$  then  $M$  is a maximal matching and  $C$  is a minimal cover.

By the previous proof, every matching  $M'$  has size  $|M'| \leq |C| = |M|$  so  $M$  is a maximal matching. Similarly, every cover  $C'$  has size  $|C'| \geq |M| = |C|$  so  $C'$  is a minimal cover.  $\square$

Note that there exists graph  $G$  such that  $|M| \neq |C|$  for a maximal matching  $M$  of  $G$  and minimal cover  $C$  of  $G$ .



Maximal matching in red and minimal cover in blue.

Let  $\nu(G)$  denote the size of a maximal matching of  $G$  and  $\tau(G)$  denote the size of a minimal cover.

**Theorem 5.22** (*Konig's Theorem*)

*In a bipartite graph, the maximum size of a matching is equal to the minimum size of a cover.*

**Proof 5.35** Of Konig's Theorem

**The X-Y Construction:** Let  $G$  be a bipartite graph with bipartition  $(A, B)$ . Let  $M$  be a matching of  $G$ .

Let  $X_0$  be the set of unsaturated vertices in  $A$ . Let  $Z$  be the set of all vertices  $v$  of  $G$  so that there is an alternating path from some  $x \in X_0$  to  $v$ . Let  $X = Z \cap A, Y = Z \cap B$ .

For each  $v \in Z$ , let  $P(v)$  be an alternating path from some  $x \in X_0$  to  $v$ . Note that since  $G$  is bipartite and all vertices in  $X_0$  are in  $A$ ,

1. If  $v \in X$  then  $P(v)$  has even length and its last edge is in  $M$  since  $v \in A$
2. If  $v \in Y$  then  $P(v)$  has odd length and its last edge is not in  $M$  since  $v \in B$ .

**Lemma:** Given  $G, A, B, X, Y$  as above

- a) There is no edge of  $G$  from  $X$  to  $B \setminus Y$ .

- b)  $C = (A \setminus X) \cup Y$  is a cover of  $G$ .
- c) There is no edge in  $M$  from  $A \setminus X$  to  $Y$ .
- d) Let  $Y_0$  be the set of unsaturated vertices in  $Y$ . Then  $|M| = |C| - |Y_0|$ .
- e) For every  $y \in Y_0$ ,  $P(y)$  is an augmenting path.

**Lemma A:** If  $xv$  is an edge with  $x \in X, v \in B \setminus Y$  then  $P(x), x$  is an alternating path from some vertex in  $X_0$  to  $v$ , contradicting  $v \notin Y$ .

**Lemma B:** Follows from Lemma A and the definition of a cover.

**Lemma C:** If  $yv$  is an edge in  $M$  and  $y \in Y, v \in A \setminus X$  then  $P(y), v$  is an alternating path from some vertex in  $X_0$  to  $v$ , contradicting  $v \notin X$ .

**Lemma D:** By Lemma A and C, every edge in  $M$  is either from  $X$  to  $Y$  or from  $A \setminus X$  to  $B \setminus Y$ . There are  $|Y| - |Y_0|$  edges of the first type and since every vertex in  $A \setminus X$  is saturated, there are  $|A \setminus X|$  edges of the second type. So the size of  $|M| = |Y| - |Y_0| + |A \setminus X| = |C| - |Y_0|$ .

**Lemma E:** Follows because both ends are unsaturated by definition.

**Proof of Konig's Theorem:** Let  $G$  be a bipartite graph with bipartition  $(A, B)$  and let  $M$  be a max matching of  $G$ . Construct  $X, X_0, Y, Y_0$  as above. Since  $M$  is maximum, it has no augmenting paths. So by Lemma E,  $Y_0 = \emptyset$ . Then  $C = (A \setminus X) \cup Y$  is a cover and by Lemma D,  $|M| = |C|$ . So  $M$  is a max matching and  $C$  is a min cover.  $\square$

## Max Bipartite Matching Algorithm

Input: Bipartite graph  $G$  with bipartition  $(A, B)$

Step 1: Let  $M$  be any matching of  $G$  (eg.  $\emptyset$ )

Step 2: Let  $\hat{X}$  be the set of unsaturated vertices in  $A$  and  $\hat{Y} = \emptyset$

Step 2a: (Grow  $\hat{Y}$ ) For each vertex  $v \in B \setminus \hat{Y}$  that is adjacent to a vertex  $u \in \hat{X}$ , add  $v$  to  $\hat{Y}$  and let  $\text{pr}(v) = u$ . (pr stands for parent)

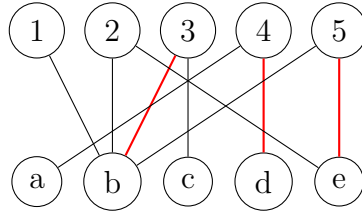
Step 2b: If  $\hat{Y}$  contains an unsaturated vertex  $y$ , then  $y, \text{pr}(y), \text{pr}(\text{pr}(y)), \dots$  is an augmenting path. Use this path to make  $M$  bigger and repeat from 1.

Step 2c: If Step 2 added no new vertex to  $\hat{Y}$ , then  $M$  is a max matching and  $C = (A \setminus \hat{X}) \cup \hat{Y}$ . Return.

Step 3: (Grow  $\hat{X}$ ) For each vertex  $u \in A \setminus \hat{X}$  that is joined by an edge of  $M$  to a vertex  $v \in \hat{Y}$ , add  $u$  to  $\hat{X}$ , set  $\text{pr}(u) = v$ . Goto 2.

**Example 5.6** Example of algorithm.

1. Start with  $\hat{X} = \{1, 2\}$ ,  $\hat{Y} = \emptyset$ .



2. Grow  $\hat{Y}$ :

$$\hat{X} = \{1, 2\}, \hat{Y} = \{b, e\}, \text{pr}(b) = 1 \text{ and } \text{pr}(e) = 2.$$

2b and 2c do not apply so continue to step 3.

3. Grow  $\hat{X}$ :

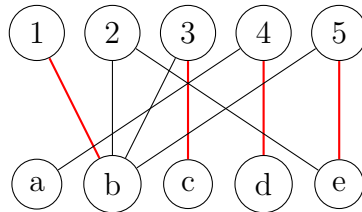
$$\hat{X} = \{1, 2, 3, 5\}, \hat{Y} = \{b, e\}, \text{pr}(3) = b \text{ and } \text{pr}(5) = e$$

4. Grow  $\hat{Y}$ :

$$\hat{X} = \{1, 2, 3, 5\}, \hat{Y} = \{b, e, c, d\}, \text{pr}(c) = 3 \text{ and } \text{pr}(d) = 3.$$

$c$  is unsaturated and  $c \in \hat{Y}$  so  $c, \text{pr}(c), \text{pr}(\text{pr}(c)), \dots = c, 3, b, 1$  is augmenting.

5. We use the path above to grow  $M$ . Set  $\hat{X} = \{2\}$  and  $\hat{Y} = \emptyset$ .



6. Grow  $\hat{Y}$ :

$$\hat{X} = \{2\}, \hat{Y} = \{b, e\}, \text{pr}(b) = 2 \text{ and } \text{pr}(e) = 2.$$

2b and 2c do not apply.

7. Grow  $\hat{X}$ :

$$\hat{X} = \{2, 1, 5\}, \hat{Y} = \{b, e\}, \text{pr}(1) = b \text{ and } \text{pr}(5) = e.$$

8. Grow  $\hat{Y}$ :

$$\hat{X} = \{2, 1, 5\}, \hat{Y} = \{b, e\}.$$

2c applies so  $M$  is a max matching. Then  $A(\setminus \hat{X}) \cup \hat{Y} = \{3, 4, b, e\}$  is a min cover.



If  $X$  is a set of vertices in a graph  $G$ , then the **neighbourhood** of  $X$ , denoted  $N(X)$  is the set of vertices of  $G$  that are adjacent to a vertex in  $X$ .

Observe that if  $X$  is a set of vertices in a graph  $G$  with  $|N(X)| < |X|$  then  $G$  has no matching saturating every vertex in  $X$ .

**Theorem 5.23** (*Hall's Theorem*)

Let  $G$  be a bipartite graph with bipartition  $(A, B)$ . Then  $G$  has a matching saturating  $A$  iff  $|N(A')| \geq |A'|$  for all  $A' \subseteq A$ .

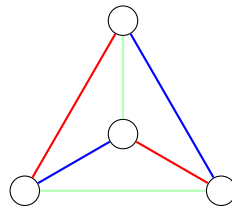
**Proof 5.36** Clearly if there exists  $A' \subseteq A$  with  $|N(A')| < |A'|$  then  $G$  has no matching saturating  $A$ .

Conversely suppose that  $|N(A')| \geq |A'|$  for all  $A' \subseteq A$ . To show that there is a matching saturating  $A$ , it suffices to show that  $A$  is a min cover (by Konig's Theorem).

Let  $C$  be a cover of  $G$ . Consider  $A \setminus C$ . Because  $C$  is a cover, every neighbour of a vertex in  $A \setminus C$  is in  $B \cap C$ . So  $N(A \setminus C) \subseteq B \cap C$ . Therefore  $|B \cap C| \geq |N(A \setminus C)| \geq |A \setminus C|$  by assumption. So  $|C| = |C \cap A| + |C \cap B| = |C \cap A| + |A \setminus C| = |A|$ . So  $A$  is a min cover.

## Edge Colouring

A  **$k$ -edge colouring** of graph  $G$  is an assignment of a colour from a set of  $k$  colours to each edge of  $G$  so that edges sharing an end get different colours.



3-edge colouring of  $K_4$

**Theorem 5.24** For  $k > 0$ , every  $k$ -regular bipartite graph has a perfect matching.

**Proof 5.37** Since the number of edges is  $k|A| = k|B|$  we have  $|A| = |B|$ .

By Hall's theorem, for a perfect matching to exist we have to show that  $|N(A')| \geq |A'|$  for all  $A' \subseteq A$ . Let  $A' \subseteq A$ . Let  $F$  be the set of edges from  $A'$  to  $N(A')$ .

Every edge with one end in  $A'$  is in  $F$  so  $|F| = k|A'|$ . Also every edge of  $F$  has one end in  $N(A')$ , so  $|F| \leq k|N(A')|$ . Then  $k|A'| = |F| \leq k|N(A')|$  so  $|A'| \leq |N(A')|$ . Hall's theorem gives us the result.  $\square$